Def (ODE) An ordinary differential equation is an equation of the form
\[ F(x, y, y', ..., y^{(n)}) = 0 \]
for \( F: \mathbb{R}^{n+2} \to \mathbb{R} \)

Def (Linearity) An ODE
\[ F(x, y, ..., y^{(n)}) = 0 \]
is **linear** if \( F(x, -): \mathbb{R}^{n+1} \to \mathbb{R} \) is a linear map.

We can write a linear ODE of the form
\[ a_n(x) y^{(n)}(x) + \ldots + a_1(x) y'(x) + a_0(x) y(x) + f(x) = 0 \]
for \( a_i: \mathbb{R} \to \mathbb{R}, \quad f: \mathbb{R} \to \mathbb{R} \)

- \( n \) is called an **order (degree)** of linear ODE
- Solutions to an ODE are often called **integral curves**
Def (Exact equations) Let $F, M, N : \mathbb{R}^2 \to \mathbb{R}$ be differentiable functions. Suppose
\[
\frac{\partial F}{\partial x} = M(x,y), \quad \frac{\partial F}{\partial y} = N(x,y).
\]
Then an ODE of the form
\[
M(x,y) + N(x,y) \frac{dy}{dx} = 0
\]
is called an exact equation (also written in the form)
\[
M(x,y) \ dx + N(x,y) \ dy = 0
\]

Def (IVP) An initial value problem is a differential equation along with an appropriate number of initial conditions (order -1)

\[\begin{align*}
\text{eg: } & 4y'' + y'y' + e^x = 0 & y(0) = 1 & y'(0) = 2 \\
\text{Separable equations} & & & \\
\frac{dy}{dx} = \frac{f(x)}{g(y)} \\
g(y) \ dy = f(x) \ dx \\
\int g(y) \ dy = \int f(x) \ dx.
\end{align*}\]
e.g. \( \frac{dy}{dx} = \frac{x}{ey} \quad y(0) = 0 \)

\[ e^y dy = x dx \]

\[ e^y = \frac{1}{2} x^2 + C \]

\[ y = \log \left( \frac{1}{2} x^2 + C \right) \]

\[ y(0) = \log C = 0 \quad C = 1 \]

**Homogeneous equations**

\[ f(x,y) \text{ is homogeneous of degree } n \text{ if} \]

\[ f(tx, ty) = t^n f(x, y) \]

\[ M(x,y) \, dx + N(x,y) \, dy = 0 \]

is homogeneous if \( M, N \) are homogeneous functions of the same degree.

e.g. \( (x^2 + y^2) \, dx - 2xy \, dy = 0 \)

\[ y = tx \implies (x^2 + t^2 x^2) \, dx - 2x^2t \, dy = 0 \]
\[ (1 + t^2) \, dx - 2t \, dt = 0 \]
\[ (1 - t^2) \, dx - 2tx \, dt = 0 \]
\[ \frac{1}{x} \, dx = \frac{2t}{1 - t^2} \, dt \]
\[ \ln |x| = -\ln |1 - t^2| + \ln c \]
\[ |x| = \frac{c}{1 - t^2} = \frac{c}{1 - (\frac{y}{x})^2} \quad c > 0 \]
\[ x^2 - y^2 = cx \quad c \neq 0 \]

**Exact equations**

\[ M(x,y) \, dx + N(x,y) \, dy = 0 \]

is exact if \[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]

\[ F_x = M(x,y) \quad F_y = N(x,y) \]

sol: \[ F(x,y) = C \]

e.g. \[ (1 - 2xy) \, dx + (4y^3 - x^2) \, dy = 0 \]
\[ F = \int M \, dx = \int (1 - 2xy) \, dx = x - x^2y + g(y) \]

\[ F_y = -x^2 + g'(y) = N = 4y^3 - x^2 \]

\[ g(y) = \int 4y^3 \, dy = y^4 \]

\[ \therefore F(x, y) = x - x^2y + y^4 = C. \]

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**Integrating factors**

\( \mu(x) \) or \( \eta(y) \)

\( x \) integrating factor

Not exact \( \rightarrow \) exact

\( \frac{M_y - N_x}{N} = f(x) \Rightarrow \mu(x) = e^{\int f(x) \, dx} \)

\( \frac{M_y - N_x}{M} = g(y) \Rightarrow \eta(y) = e^{\int g(y) \, dy} \)

\( \text{e.q.} \quad (xy + x - 1) \, dx + x^2 \, dy = 0 \)

\( \frac{M_y - N_x}{N} = -\frac{1}{x} \Rightarrow \mu(x) = e^{\int -\frac{1}{x} \, dx} = \frac{1}{x} \)
\[ \frac{d}{dx} \left( y + \frac{1}{x} \right) \, dx + x \, dy = 0 \]

\[ F = \int \left( y + \frac{1}{x} \right) \, dx = xy + \frac{1}{2} \ln|x| + h(y) \]

\[ F_y = N = x + h'(y) = x \]

\[ F(x, y) = xy + \frac{1}{2} \ln|x| = C. \]

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1st order ODE

\[ \frac{dy}{dx} + P(x) \cdot y = Q(x) \]

\[ \mu(x) = e^{\int P(x) \, dx} \]

\[ \mu y' + \mu Py = \mu Q \]

\[ \Rightarrow (\mu y)' = \mu Q \quad \text{(} \mu'y = \mu P \mu y \text{)} \]

\[ \Rightarrow y = \frac{1}{\mu} \int \mu \cdot Q \, dx \]

\[ e.g. \quad \frac{dy}{dx} = 5x - \frac{3y}{x}, \quad y(1)=2 \]

\[ \mu = e^{\int \frac{3}{x} \, dx} = x^3 \]
\[ x^3 y = \int 5x^4 \, dx = x^5 + c \]
\[ y = x^2 + \frac{c}{x^2} \]
\[ y(1) = 1 + c \quad c = 1 \]

High order linear ODE w/ const coef:

\[ ay'' + by' + cy = d(x) \quad \text{--- (\#)} \]

\[ y = y_p + y_h \]

\[ y_p : \text{particular solution of (\#)} \]
\[ y_h : \text{general solutions of } ay'' + by' + cy = 0 \]

Characteristic eq: \[ at^2 + bt + c = 0 \quad \text{--- (\#)} \]

1. roots of (\#) are real and distinct:
\[ t = m_1, m_2 \]
\[ y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x} \]

2. roots are real and identical: \[ t = m \]
\[ y_h = c_1 e^{m x} + c_2 x e^{m x} \]
3. roots are not real \( t = \alpha + \beta i \)

\[ y_h = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) \]