Permutations and combinations

Permutation Choose k things from n, with regard to order

$$P(n,k) = \frac{n!}{(n-k)!}$$

Combinations Choose k things from n, without regard to order

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

binomial coefficients

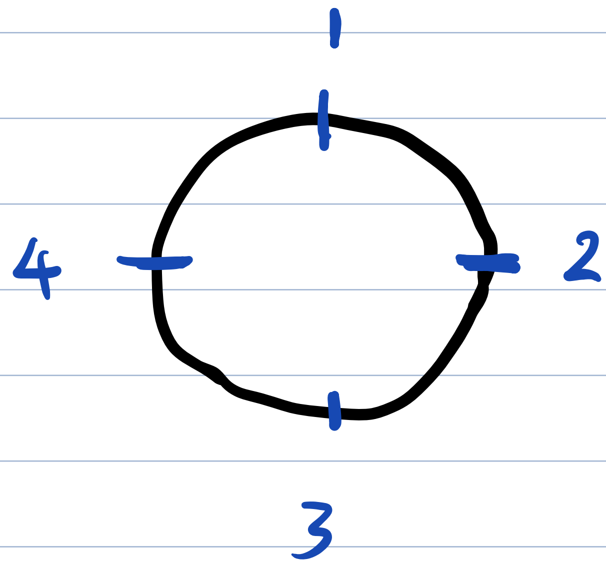
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

e.g $2^n = \sum_{k=0}^{n} \binom{n}{k}$, $\quad 0 = \sum (-1)^k \binom{n}{k}$

Prop $\binom{n}{k} = \binom{n}{n-k}$

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Circular permutation



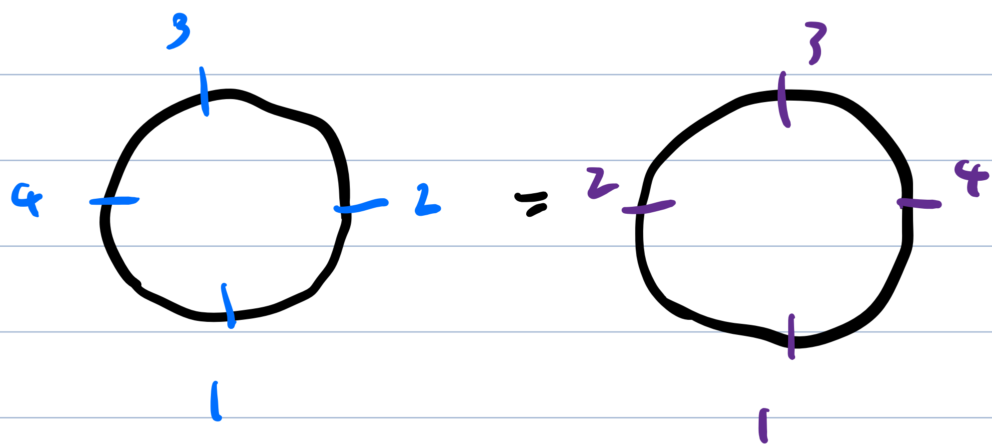
Arrange n things on a round table w/ regard to order.

$$\frac{n!}{n} = (n-1)!$$

Necklace permutation

Arrange n beads to form a necklace.

$$\frac{(n-1)!}{2}$$

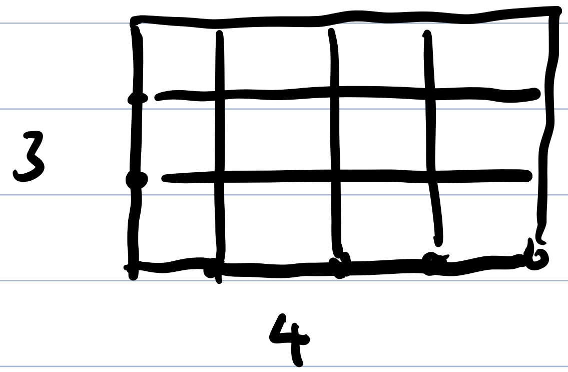


Permutation w/ the same things

$\underbrace{aaa}_{n_1}\ \underbrace{bbb}_{n_2}\ \underbrace{cccc}_{n_3}\ \cdots \qquad n = \sum_{i=1}^{k} n_i$

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Shortest paths



3 ups 4 rights

uuurrrr, ...

$$\frac{7!}{3!\,4!}$$

Permutation w/ repetitions allowed

n things & Choose k : n^k

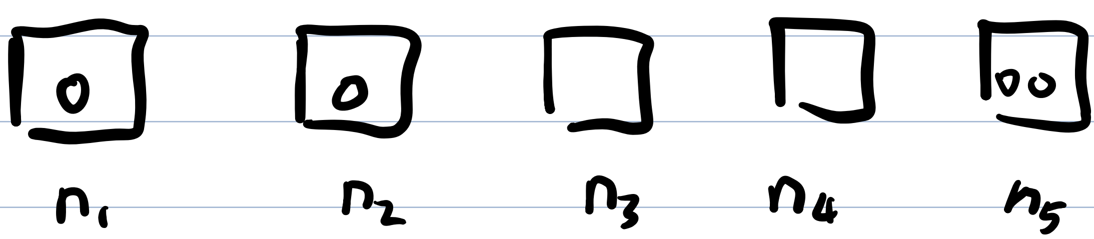
Combination w/ repetitions allowed

$$H(n,k) = \binom{n+k-1}{k}$$

e.g. How many ways can we write the number 4 as the sum of 5 nonnegative integers ?

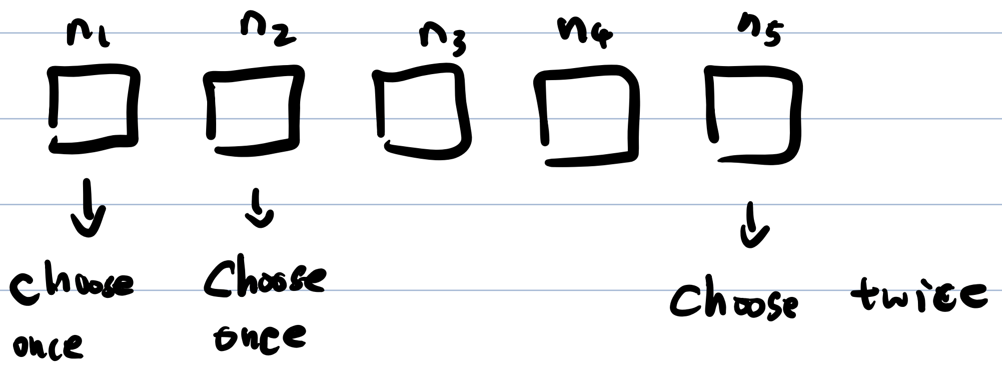
sol) $n_1 + n_2 + n_3 + n_4 + n_5 = 4$

① We need to distribute 4 balls to 5 boxes



$\rightarrow \quad 1+1+0+0+2=4$

② We choose 4 boxes out of 5 w/ repetitions



$$H(5,4) = \binom{5+4-1}{4} = \binom{8}{4} = \frac{8!}{4!4!} = 70.$$

Probability

$P(A)$: Probability of events A.
event

$$P(A \cup B) = P(A) + P(B) + P(A \cap B)$$

Events A and B are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

(one does not influence the other)
change the probability

Events A and B are mutually exclusive if

$$P(A \cap B) = 0$$

$$\left(P(A \cup B) = P(A) + P(B) \right)$$

e.g Throw two dice twice.

A: event that the first toss is 7 or 11

B: event that the second toss is 11

$P(A \text{ or } B) = ?$

sol) $A = \{ (1,6)\ (2\ 5)\ (3\ 4)\ (4\ 3)\ (5\ 2)\ (6\ 1)$

$(6\ 5)\ (5\ 6) \}$

$B = \{ (5\ 6)\ (6\ 5) \}$

A, B: independent events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{8}{36} + \frac{2}{36} - \frac{8}{36} \cdot \frac{2}{36}$$

$$= \frac{43}{162}.$$

Conditional probability

$P(A|B)$: the probalitiy of the event A will occur given that the event B has already occured.

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

Random variables : A function $X : S \to \mathbb{R}$
samples

e.g Toss a coin : $S = \{H, T\}$

$X : S \to \mathbb{R}$

$X(H) = 0 \quad X(T) = 1$

$P(X=0) = \frac{1}{2} \quad P(X=1) = 1$

distribution of X : $F_X(t) := P(X \leq t)$

density of X : $f_X(t)$

$$\left[\begin{array}{l} F_X(t) = \int_{-\infty}^{t} f_X(t)\,dt \\ \int_{-\infty}^{\infty} f_X(t)\,dt = 1 \end{array}\right.$$

expectation (mean) of X : $E(X) = \mu(X) := \int_{-\infty}^{\infty} t f_X(t)\, dt$

variance of X : $Var(X) = \sigma^2(X) := \int_{-\infty}^{\infty} (t - \mu(X))^2 f_X(t)\, dt$

standard deviation of X : $\sigma(X)$

- $Var(X) = E(X^2) - E(X)^2$

Markov inequality :

$$P(|X| \geq a) \leq \frac{E(X)}{a} \quad \text{for } a > 0$$

Chebyshev inequality :

$$P(|X - \mu| \geq K\sigma) \leq \frac{1}{K^2} \quad \text{for } K > 0$$