Def Let X be a set and τ a collection of subsets of X. We say τ is a topology on X if

1) $\emptyset, X \in \tau$
2) $U_1, \cdots, U_n \in \tau \Rightarrow \bigcap_{i=1}^{n} U_i \in \tau$
3) $\{U_i\}_{i\in I} \subset \tau \Rightarrow \bigcup_{i\in I} U_i \in \tau$

We say an element in τ is an open set.

e.g trivial topology : $\tau = \{\emptyset, X\}$

discrete topology : $\tau = \mathcal{P}(X)$ (power set of X)

(i.e. any $A \subset X \Rightarrow A \in \tau$)

Def Let (X, τ) be a topological space. Then for any $U \in \tau$, U^c is called a closed set

Def Let $(X.\tau)$ be a topological space and $A \subset X$. The interior of A is the largest open set contained in X. The closure of A is the smallest closed set containing A.

$$\mathring{A} := \mathrm{int}(A) := \bigcup_{\substack{U\in\tau \\ U\subset A}} U \qquad \bar{A} := \mathrm{cl}(A) := \bigcap_{\substack{S^c\in\tau \\ A\subset S}} S$$

Def Let X be a set and β a collection of subsets of X s.t.

1) $X = \bigcup_{B \in \beta} B$

2) If $B_1, B_2 \in \beta$, then for $x \in B_1 \cap B_2$, there is $B_3 \in \beta$ s.t $x \in B_3$ and $B_3 \subset B_1 \cap B_2$

Then $\tau := \{ U : U = \bigcup_{i \in I} B_i$ for any subcollection $\{B_i\}_{i \in I} \subset \beta \}$

is a topology on X and β is called a basis of τ (and τ is called a topoloy generated by β)

e.g The standard topology on $\mathbb{R}$ is a topology generated by a basis $\{ (a,b) : a, b \in \mathbb{R} \}$

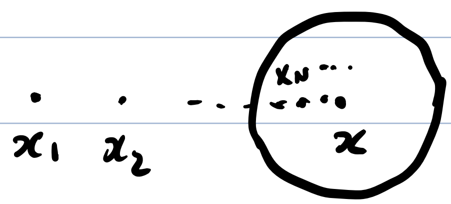
Def Let (X, τ) be a topological space and $Y \subset X$. Then

$$\tau_Y := \{ Y \cap U : U \in \tau \}$$

is a topology on Y and called the subspace topology on Y

Def Let (X,τ) and (Y,σ) be topological spaces. Then $\beta := \tau \times \sigma = \{U \times V : U \in \tau, V \in \sigma\}$ is a basis for a topology on $X \times Y$.
This topology is called the product topology of (X,τ) and (Y,σ)

Def Let (X,τ) be a topological space and $(x_n)_{n=1}^{\infty}$ a sequence in X. We say x_n converges to x if for any open set $U \subset X$ containing x, there is $N > 0$ s.t $x_n \in U$ for $n > N$.



e.g • In the trivial topology $(X, \tau = \{\phi, X\})$, any sequence converges to any point.

• In the discrete topology, any sequence does not converge.

Def A topological space (X,τ) is called Hausdorff if for any $x, y \in X$, there are open sets U_x and U_y
$(x \neq y)$

containing x and y respectively, s.t. $U_x \cap U_y = \phi$.

Prop In a Hausdorff space, if there is a limit of a sequence, it is unique.

e.g The trivial topology is not Hausdorff
The discrete topology is Hausdorff.

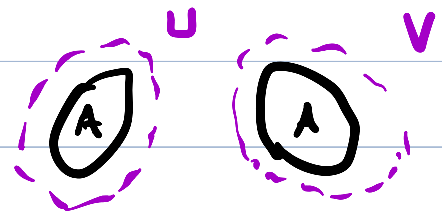
Def Let (X, τ) be a topological space and $A \subset X$. A limit point (= cluster point, accumulation point) of A is a point $x \in X$ s.t any open set containing x contains a point in A which is different from x

e.g $(\mathbb{R}, \tau_{std})$, standard topology, $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, 0 is the limit pt of A.

Def Let (X, τ) be a topological space. A set $A \subset X$ is compact if for any collections of open sets $\{U_i\}_{i \in I}$ s.t. $A \subset \bigcup_{i \in I} U_i$, there is a finite subcollection $\{U_{i_1}, \dots, U_{i_n}\} \subset \{U_i\}$ s.t $A \subset \bigcup_{k=1}^{n} U_{i_k}$

Prop In $(\mathbb{R}^n, \tau_{std})$, a set $A \subset \mathbb{R}^n$ is compact if and only if A is closed and bounded.

Def Let (X, σ) be a topological space. A set $A \subset X$ is disconnected if there is a two open sets $U, V \subset X$ s.t $U \cap V = \emptyset$ $U \cup V = A$. If A is not disconnected we call it connected



Def Suppose (X, τ) and (Y, σ) are topological spaces. Then $f: X \to Y$ is continuous if

$f^{-1}(U)$ is open in X for any open set U in Y

Prop Suppose $f: (X, \tau) \to (Y, \sigma)$ is a continuous function. Then

1) for any closed set $C \subset Y$ $f^{-1}(C)$ is closed
2) for any compact set $A \subset X$ $f(A)$ is compact
3) for any connected set $A \subset X$ $f(A)$ is connected
4) If $f, g: X \to Y$ are continuous, then $(f, g): (X \times X) \to (Y \times Y)$ is also continuous.

e.g (topologist's sine curve)

$C = \{(0,y) : y \in [-1,1]\} \cup \{(x, \sin\frac{1}{x}) : x \in (0,\infty)\}$

$\underbrace{\{(0,y) : y \in [-1,1]\}}_{A}$ $\quad$ $\underbrace{\{(x, \sin\frac{1}{x}) : x \in (0,\infty)\}}_{B}$

C is connected :

① A is connected (exercise)

② B is connected : B is the image of

$f : (0,\infty) \to \mathbb{R}\times\mathbb{R}$ $\quad f(x) = (x, \sin\frac{1}{x})$

Since f is continuous and $(0,\infty)$ is connected,
B is connected.

③ $A \cup B$ is connected : Since any open set
containing A intersects B (exercise),
there is no open sets U, V s.t. $A \subset U$ $B \subset V$
and $U \cap V = \emptyset$. $\therefore$ $A \cup B$ is connected.