Def. Let \( X \) be a set and \( \mathcal{Z} \) a collection of subsets of \( X \). We say \( \mathcal{Z} \) is a topology on \( X \) if

1) \( \emptyset, X \in \mathcal{Z} \)
2) \( \bigcup_{i \in I} U_i \in \mathcal{Z} \) if \( U_i \in \mathcal{Z} \) for all \( i \in I \)
3) \( \bigcap_{i \in I} U_i \in \mathcal{Z} \) if \( U_i \in \mathcal{Z} \) for all \( i \in I \)

We say an element in \( \mathcal{Z} \) is an open set.

E.g. trivial topology: \( \mathcal{Z} = \{ \emptyset, X \} \)
discrete topology: \( \mathcal{Z} = \mathcal{P}(X) \) (power set of \( X \))

Def. Let \((X, \mathcal{Z})\) be a topological space. Then for any \( U \in \mathcal{Z} \), \( U^c \) is called a closed set.

Def. Let \((X, \mathcal{Z})\) be a topological space and \( A \subset X \).
The interior of \( A \) is the largest open set contained in \( X \). The closure of \( A \) is the smallest closed set containing \( A \).

\[
\overline{A} := \text{cl}(A) := \bigcap_{S \in \mathcal{Z}, S \supseteq A} S \\
\text{int}(A) = \bigcup_{U \in \mathcal{Z}, U \subseteq A} U
\]
Def: Let \( X \) be a set and \( \beta \) a collection of subsets of \( X \) s.t.

1) \( X = \bigcup_{B \in \beta} B \)

2) If \( B_1, B_2 \in \beta \), then for \( x \in B_1 \cap B_2 \), there is \( B_3 \in \beta \) s.t. \( x \in B_3 \) and \( B_3 \subset B_1 \cap B_2 \)

Then \( \tau = \{ U : U = \bigcup_{i \in I} B_i \text{ for any subcollection } \{ B_i : i \in I \} \subset \beta \} \) is a topology on \( X \) and \( \beta \) is called a basis of \( \tau \) (and \( \tau \) is called a topology generated by \( \beta \)).

E.g. The standard topology on \( \mathbb{R} \) is a topology generated by a basis \( \{ (a,b) : a, b \in \mathbb{R} \} \)

Def: Let \((X, \tau)\) be a topological space and \( Y \subset X \). Then

\[ \tau_Y = \{ Y \cap U : U \in \tau \} \]

is a topology on \( Y \) and called the subspace topology on \( Y \).
Def Let \((X,\tau)\) and \((Y,\sigma)\) be topological spaces. Then \(\beta := \mathbb{R} \times \mathbb{R} = \{u \times v : u \in \mathbb{R}, v \in \mathbb{R}\}\) is a basis for a topology on \(X \times Y\). This topology is called the product topology of \((X,\tau)\) and \((Y,\sigma)\).

Def Let \((X,\tau)\) be a topological space and \((x_n)_{n=1}^{\infty}\) a sequence in \(X\). We say \(x_n\) converges to \(x\) if for any open set \(U \subset X\) containing \(x\), there is \(N \geq 0\) s.t. \(x_n \in U\) for \(n \geq N\).

- \(x_1, x_2, \ldots, x_n\)
- \(x\)

E.g. In the trivial topology \((X,\tau) = \{\emptyset, X\}\), any sequence converges to any point.

- In the discrete topology, any sequence does not converge.

Def A topological space \((X,\tau)\) is called Hausdorff if for any \(x,y \in X\), there are open sets \(U_x\) and \(U_y\) with \(x \in U_x\) and \(y \in U_y\) such that \(U_x \cap U_y = \emptyset\).
containing $x$ and $y$ respectively, s.t. $U_x \cap U_y = \emptyset$.

**Prop** In a Hausdorff space, if there is a limit of a sequence, it is unique.

e.g. The trivial topology is not Hausdorff.
The discrete topology is Hausdorff.

**Def** Let $(X, \tau)$ be a topological space and $A \subseteq X$. A limit point (clustering point, accumulation point) of $A$ is a point $x \in X$ s.t. any open set containing $x$ contains a point in $A$ which is different from $x$.

- Standard topology

e.g. $(\mathbb{R}, \text{standard})$, $A = \{ \frac{1}{n} : n \in \mathbb{N} \}$, $0$ is the limit pt of $A$.

**Def** Let $(X, \tau)$ be a topological space. A set $A \subseteq X$ is compact if for any collections of open sets $\{ U_i \}_{i \in I}$ s.t. $A \subseteq U_i$, there is a finite subcollection $\{ U_{i_1}, \ldots, U_{i_n} \} \subseteq \{ U_i \}$ s.t. $A \subseteq \bigcup_{k=1}^{n} U_{i_k}$. 


Prop In $\mathbb{R}^n$, a set $A \subseteq \mathbb{R}^n$ is compact if and only if $A$ is closed and bounded.

Def Let $(X, \tau)$ be a topological space. A set $A \subseteq X$ is disconnected if there is a two open sets $U, V \subseteq X$ s.t. $U \cap V = \emptyset$ and $U \cup V = A$. If $A$ is not disconnected we call it connected.

Def Suppose $(X, \tau)$ and $(Y, \sigma)$ are topological spaces. Then $f : X \to Y$ is continuous if

$f^{-1}(U)$ is open in $X$ for any open set $U$ in $Y$.

Prop Suppose $f : (X, \tau) \to (Y, \sigma)$ is a continuous function. Then

1) for any closed set $C \subseteq Y$, $f^{-1}(C)$ is closed.
2) for any compact set $A \subseteq X$, $f(A)$ is compact.
3) for any connected set $A \subseteq X$, $f(A)$ is connected.
4) If $f, g : X \to Y$ are continuous, then $(f \circ g) : (X \times X) \to (Y \times Y)$ is also continuous.
e.g. (topologist's sine curve)

\[ C = \{ (0, y) : y \in [-1, 1] \} \cup \{ (x, \sin \frac{1}{x}) : x \in (0, \infty) \} \]

\( A \quad B \)

\( C \) is connected:

1. \( A \) is connected (exercise)
2. \( B \) is connected; \( B \) is the image of
   \[ f : (0, \infty) \to \mathbb{R} \times \mathbb{R} \quad f(x) = (x, \sin \frac{1}{x}) \]
   Since \( f \) is continuous and \((0, \infty)\) is connected, \( B \) is connected.
3. \( A \cup B \) is connected: Since any open set containing \( A \) intersects \( B \) (exercise), there is no open sets \( U, V \) s.t. \( A \cup U \subset B \subset V \) and \( U \cap V = \emptyset \), \( \therefore A \cup B \) is connected.