GRE Math Subject Prep Course: Abstract Algebra

July 12, 2021

1. (Exam VI Prob 28)[1] Let the binary operation $\circ$ be defined on all integers by $a \circ b = a + b + ab$. Which of the following statements are true?

   I. $\circ$ is associative.

   II. $\circ$ is commutative.

   III. For every integer a, there is an integer inverse a^{-1} such that $a \circ a^{-1} = 1$.

   (A) I only (B) II only (C) I and II only

   (D) III only (E) none of the above

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2. (Chapter 6 Prob 10)[2] If $S = \{a \in \mathbb{R}^+ : a \neq 1\}$, with the binary operation $\circ$ defined by the equation $a \circ b = a^{\log b}$ (where $\log b = \log_e b$), then $(S, \circ)$ is a group. What is the inverse of $a \in S$?

   (A) $\dfrac{1}{e \log a}$ (B) $\dfrac{e}{\log a}$ (C) $e^{-\log a}$

   (D) $e^{\log(1/a)}$ (E) $e^{1/\log a}$

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3. (Exam V Prob 36) Which of the following are groups?

   I. All integers under subtraction

   II. All non-zero real numbers under division

   III. All even integers under addition

   IV. All integers which are multiples of 13 under addition

   (A) I and II only (B) II and III only (C) III only

   (D) IV only (E) III and IV only

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4. (Chapter 6 Prob 8) If G is an Abelian group of order 12, then G must have a subgroup of all the following orders EXCEPT

   (A) 2 (B) 3 (C) 4

   (D) 6 (E) 12

[1] The problems with "Exam I" – "Exam VI" are taken from the REA book "The Best Test Preparation for the GRE Mathematics Test", 4th edition.

[2] The problems with "Chapter *" are taken from "Cracking the GRE Mathematics Test", 4th Edition.

5. (Exam I Prob 5) The number of generators of the cyclic group of order 8 is

(A) 6 (B) 4 (C) 3

(D) 2 (E) 1

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6. Which one of the following groups is cyclic?

(A) $\mathbf{Z}_2 \times \mathbf{Z}_4$ (B) $\mathbf{Z}_2 \times \mathbf{Z}_6$ (C) $\mathbf{Z}_3 \times \mathbf{Z}_4$

(D) $\mathbf{Z}_3 \times \mathbf{Z}_6$ (E) $\mathbf{Z}_4 \times \mathbf{Z}_6$

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7. (Exam III Prob 18) Let G be a group, and $a \in G$ is some fixed element. The mapping $\phi : G \to G$ is given by $\phi(g) = a^2ga^2$ for every element $\phi \in G$. Then ϕ is a homomorphism if

(A) $a^4 = e$ (B) $a^3 = e$ (C) $ag = ga, \forall g \in G$

(D) G is abelian (E) G is finite

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8. (Exam I Prob 36) Up to isomorphism, how many Abelian groups are there of order 36?

(A) 1 (B) 4 (C) 9

(D) 12 (E) 18

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9. (Exam VI Prob 24) Let $U = \{0, 1, c\}$ be a ring with three elements (1 is the unity). Which statements are true?

I. $1 + 1 + 1 = 0$
II. $1 + 1 = c$
III. $c^2 = 1$

(A) I only (B) II only (C) I and II only

(D) II and III only (E) I, II and III

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10. (Exam III Prob 6) Find the characteristic of the ring $\mathbf{Z}_2 \oplus \mathbf{Z}_3$.
(Note: the *characteristic* of a ring R is the smallest positive number n such that $\underbrace{1 + 1 + ... + 1}_{n \text{ times}} = 0$, where 1 denote the multiplicative identity element of the ring R.)

(A) 0 (B) 6 (C) 3

(D) 4 (E) 2

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11. (Exam II Prob 62) Let R be a ring, and let $x \neq 0$ be a fixed element in R. Which of the following is a subring of R?

   (A) $\{r \in R : xr = 0\}$

   (B) $\{r \in R : r^{-1} \text{ exists in } R\}$

   (C) $\{x^n : n = 1, 2, 3 \ldots\}$

   (D) $\{nx : n \text{ is an integer}\}$

   (E) Both (A) and (D)

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12. (Exam II Prob 27) Let R be a ring such that $x^2 = x$ for each $x \in R$. Which of the following must be true?

   (A) $x = -x$ for all $x \in R$ (B) R is commutative (C) $xy + yx = 0,\ \forall x, y \in R$

   (D) Both (A) and (C) (E) (A),(B) and (C)

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13. (Chapter 6 Prob 14) Let H be the set of all group homomorphisms $\phi : \mathbf{Z}_3 \to \mathbf{Z}_6$. How many functions does H contain?

   (A) 1 (B) 2 (C) 3

   (D) 4 (E) 6

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14. (Chapter 6 Prob 20) Which of the following are subfields of $\mathbb{C}$?

   I. $K_1 = \{a + b\sqrt{2/3} : a, b \in \mathbb{Q}\}$

   II. $K_2 = \{a + b\sqrt{2} : a, b \in \mathbb{Q} \text{ and } ab < \sqrt{2}\}$

   III. $K_3 = \{a + bi : a, b \in \mathbb{Z} \text{ and } i = \sqrt{-1}\}$

   (Note: $\mathbb{C}$ is the set of complex numbers, $\mathbb{Q}$ is the set of rational numbers, $\mathbb{Z}$ is the set of integer numbers)

   (A) I only (B) I and II only (C) III only

   (D) I and III only (E) None of them

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15. (Exam II Prob 41) In the finite field, $\mathbf{Z}_{17}$, the multiplicative inverse of 10 is

   (A) 13 (B) 12 (C) 11

   (D) 9 (E) 7

16. (Week 5 Prob 9) Suppose that a group has an element of order 7 but no element which is its own inverse (other than the identity). Which of the following is a possible order for this group?

(A) 27 (B) 28 (C) 35

(D) 37 (E) 42

Answer: CEED BCAB EBAE CABC