1. (Exam VI Prob 28)\(^1\) Let the binary operation \(\circ\) be defined on all integers by \(a \circ b = a + b + ab\). Which of the following statements are true?

   I. \(\circ\) is associative.
   II. \(\circ\) is commutative.
   III. For every integer \(a\), there is an integer inverse \(a^{-1}\) such that \(a \circ a^{-1} = 1\).

   (A) I only  (B) II only  (C) I and II only  
   (D) III only  (E) none of the above

2. (Chapter 6 Prob 10)\(^2\) If \(S = \{a \in \mathbb{R}^+ : a \neq 1\}\), with the binary operation \(\circ\) defined by the equation \(a \circ b = a^{\log b}\) (where \(\log b = \log_e b\)), then \((S, \circ)\) is a group. What is the inverse of \(a \in S\)?

   (A) \(\frac{1}{e \log a}\)  (B) \(\frac{e}{\log a}\)  (C) \(e^{-\log a}\)  
   (D) \(e^{\log(1/a)}\)  (E) \(e^{1/\log a}\)

3. (Exam V Prob 36) Which of the following are groups?

   I. All integers under subtraction  
   II. All non-zero real numbers under division  
   III. All even integers under addition  
   IV. All integers which are multiples of 13 under addition

   (A) I and II only  (B) II and III only  (C) III only  
   (D) IV only  (E) III and IV only

4. (Chapter 6 Prob 8) If \(G\) is an Abelian group of order 12, then \(G\) must have a subgroup of all the following orders EXCEPT

   (A) 2  (B) 3  (C) 4  
   (D) 6  (E) 12

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\(^2\)The problems with “Chapter *” are taken from “Cracking the GRE Mathematics Test”, 4th Edition.
5. (Exam I Prob 5) The number of generators of the cyclic group of order 8 is
   (A) 6  (B) 4  (C) 3
   (D) 2  (E) 1

6. Which one of the following groups is cyclic?
   (A) $\mathbb{Z}_2 \times \mathbb{Z}_4$  (B) $\mathbb{Z}_2 \times \mathbb{Z}_6$  (C) $\mathbb{Z}_3 \times \mathbb{Z}_4$
   (D) $\mathbb{Z}_3 \times \mathbb{Z}_6$  (E) $\mathbb{Z}_4 \times \mathbb{Z}_6$

7. (Exam III Prob 18) Let $G$ be a group, and $a \in G$ is some fixed element. The mapping
   $\phi: G \rightarrow G$ is given by $\phi(g) = a^2 ga^2$ for every element $\phi \in G$. Then $\phi$ is a homomorphism if
   (A) $a^4 = e$  (B) $a^3 = e$  (C) $ag = ga, \forall g \in G$
   (D) $G$ is abelian  (E) $G$ is finite

8. (Exam I Prob 36) Up to isomorphism, how many Abelian groups are there of order 36?
   (A) 1  (B) 4  (C) 9
   (D) 12  (E) 18

9. (Exam VI Prob 24) Let $U = \{0, 1, c\}$ be a ring with three elements (1 is the unity). Which
   statements are true?
   I. $1 + 1 + 1 = 0$
   II. $1 + 1 = c$
   III. $c^2 = 1$
   (A) I only  (B) II only  (C) I and II only
   (D) II and III only  (E) I, II and III

10. (Exam III Prob 6) Find the characteristic of the ring $\mathbb{Z}_2 \oplus \mathbb{Z}_3$.
    (Note: the characteristic of a ring $R$ is the smallest positive number $n$ such that
    $1 + 1 + \ldots + 1 = 0$, where 1 denote the multiplicative identity element of the ring $R$.)
    (A) 0  (B) 6  (C) 3
     (D) 4  (E) 2
11. (Exam II Prob 62) Let $R$ be a ring, and let $x \neq 0$ be a fixed element in $R$. Which of the following is a subring of $R$?

(A) $\{r \in R : xr = 0\}$
(B) $\{r \in R : r^{-1} \text{ exists in } R\}$
(C) $\{x^n : n = 1, 2, 3 \ldots\}$
(D) $\{nx : n \text{ is an integer}\}$
(E) Both (A) and (D)

12. (Exam II Prob 27) Let $R$ be a ring such that $x^2 = x$ for each $x \in R$. Which of the following must be true?

(A) $x = -x$ for all $x \in R$  
(B) $R$ is commutative  
(C) $xy + yx = 0$, $\forall x, y \in R$

(D) Both (A) and (C)  
(E) (A),(B) and (C)

13. (Chapter 6 Prob 14) Let $H$ be the set of all group homomorphisms $\phi : \mathbb{Z}_3 \to \mathbb{Z}_6$. How many functions does $H$ contain?

(A) 1  
(B) 2  
(C) 3

(D) 4  
(E) 6

14. (Chapter 6 Prob 20) Which of the following are subfields of $\mathbb{C}$?

I. $K_1 = \{a + b\sqrt{2}/3 : a, b \in \mathbb{Q}\}$
II. $K_2 = \{a + b\sqrt{2} : a, b \in \mathbb{Q} \text{ and } ab < \sqrt{2}\}$
III. $K_3 = \{a + bi : a, b \in \mathbb{Z} \text{ and } i = \sqrt{-1}\}$

(Note: $\mathbb{C}$ is the set of complex numbers, $\mathbb{Q}$ is the set of rational numbers, $\mathbb{Z}$ is the set of integer numbers)

(A) I only  
(B) I and II only  
(C) III only

(D) I and III only  
(E) None of them

15. (Exam II Prob 41) In the finite field, $\mathbb{Z}_{17}$, the multiplicative inverse of 10 is

(A) 13  
(B) 12  
(C) 11

(D) 9  
(E) 7
16. (Week 5 Prob 9) Suppose that a group has an element of order 7 but no element which is its own inverse (other than the identity). Which of the following is a possible order for this group?

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<td>(A) 27</td>
<td>(B) 28</td>
<td>(C) 35</td>
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<td>(D) 37</td>
<td>(E) 42</td>
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Answer: CEED BCAB EBAE CABC