ON NEARLY LEFSCHETZ FIBRATIONS AND SPINAL OPEN BOOKS

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ABSTRACT. We survey the recent development of spinal open books and nearly Lefschetz fibrations. The focus is on explaining past and new resules regarding the detection of symplectic fillability and classification of symplectic fillings of contact 3-manifolds in light of the new machinery. We give many examples and a list of open questions. This can be considered as an additional chapter to the lecture notes by Ozbacgi [Ozb15] and Wendl [Wen18].

1. INTRODUCTION

1.1. **Overview and definitions.** In this survey article, we provide a user's guide to the series of papers [LHMW18, LHMW20, MRW25], focusing on how to apply these results to classify and study symplectic fillings of contact 3-manifolds supported by planar uniform spinal open books.

We first recall various types of symplectic fillings.

Definition 1.1. A symplectic 4-manifold (W, ω) is said to be a **weak symplectic filling** of the contact 3-manifold (M, ξ), if M is the oriented boundary of W and $\omega|_\xi > 0$.

Definition 1.2. A symplectic 4-manifold (W, ω) is said to be a **strong symplectic filling** of the contact 3-manifold (M, ξ) if M is the oriented boundary of W, and there exists a Liouville vector field X near ∂W, i.e., $\mathcal{L}_X \omega = \omega$, and $\ker(\iota_X \omega) = \xi$.

We sometimes refer to strong symplectic fillings as symplectic fillings.

Definition 1.3. A symplectic 4-manifold (W, ω) is said to be an **exact symplectic filling (or Liouville filling)** of the contact 3-manifold (M, ξ) if M is the oriented boundary of W, there exists a global Liouville vector field X on W, i.e., $\mathcal{L}_X \omega = \omega$, and $\ker(\iota_X \omega) = \xi$.

Definition 1.4. An exact filling (W, ω) of (M, ξ) is said to be a **Weinstein filling** if the Liouville vector field X is gradient-like for a Morse function f on W, i.e., $X \cdot \nabla f > 0$ away from the singularities of X, and these singularities coincide with critical points of f. By [CE12] this is equivalent to a **Stein filling**. In light of this equivalence, in this note we will generally refer to both Weinstein and Stein fillings as Stein fillings.

It is known that weak fillability implies tightness [Eli91]. Furthermore, the preceding definitions imply the following hierarchy among tight contact structures:

$$\text{Stein} \subsetneq \text{Exact} \subsetneq \text{Strong} \subsetneq \text{Weak} \subsetneq \text{Tight}$$

It is also known that this hierarchy is proper. That is, there exist contact structures that are:

- exactly fillable but not Stein fillable [Bow12]
- strongly fillable but not exactly fillable [Ghi05, Min22]
- weakly fillable but not strongly fillable [Gir94, Eli96, Gay06]
- tight but not weakly fillable [EH02, Chr21]

However, in the case of contact 3-manifolds supported by planar open book decompositions, a remarkable simplification occurs: Wendl [Wen10] proves that all minimal (weak, strong, and exact) fillings are symplectically deformation equivalent to Stein fillings. Furthermore, these fillings are supported by Lefschetz fibrations over a disk.

Theorem 1.5 (Wendl [Wen10])**.** *Suppose a contact* 3*-manifold* (M, ξ) *is supported by a planar open book* (P, ϕ)*. Then, any minimal symplectic filling of* (M, ξ) *can be foliated by* J*-holomorphic curves that are diffeomorphic to* P *and admits a positive allowable Lefschetz fibration structure that induces the open book* (P, ϕ) *on its boundary.*

Moreover, there is a one-to-one correspondence between the deformation equivalence classes of the fillings and the set of positive allowable Lefschetz fibrations up to fiber-preserving diffeomorphism.

Definition 1.6. Let P be a compact oriented surface and $\phi \in \mathrm{Mod}_\partial(P)$. A **positive admissible factorization** of ϕ is a representative written as a product of positive Dehn twists along homologically essential curves on P.

Combined with the fact that two Hurwitz equivalent positive admissible factorizations of the monodromy induce equivalent symplectic Lefschetz fibrations, Theorem 1.5 *almost* reduces the classification of symplectic filling to a monodromy factorization problem.

Remark 1.7. Theorem 1.5 does not imply that there is a one-to-one correspondence between the set of minimal symplectic fillings and the set of positive admissible factorizations. Two different factorizations may induce equivalent symplectic fillings.

Theorem 1.5 applies to a large class of contact 3-manifolds, which are supported by planar open books – simply referred to as **planar contact** 3**-manifolds**. This monodromy factorization technique has been used to classify the symplectic fillings of some contact structures on lens spaces [PVHM10] and other large classes of planar contact 3-manifolds [Kal15, KL16]. However, since the monodromy factorization problem is difficult in general, the classification of symplectic fillings for lens spaces, in full generality, has been achieved using different techniques. For example, for universally tight contact structures on lens spaces, Lisca [Lis08] used the complements of divisor configurations in rational surfaces, and in the virtually overtwisted case, Christian–Li and Etnyre–Roy [CL23, ER21] used the "mixed torus" technique [CM18].

A major limitation of Theorem 1.5 is that there exist contact 3-manifolds that cannot be supported by planar open books. The first obstruction was found by Etnyre [Etn04b].

Theorem 1.8 ([Etn04a, Theorem 4.1])**.** *Let* (Y, ξ) *be a contact* 3*-manifold supported by a planar open book, then any symplectic filling of* (Y, ξ) *is negative definite.*

According to this theorem, the unique Stein fillable contact structure on $\mathbb{T}^3$ does not admit a planar open book decomposition, since its Stein filling $\mathbb{T}^2 \times \mathbb{D}^2$, the disk cotangent bundle of the torus, is indefinite.

Nevertheless, Wendl [Wen10] classifies the symplectic fillings of this contact structure on $\mathbb{T}^3$ using techniques similar to the proof of Theorem 1.5, by utilizing a topological decomposition similar to a planar open book. This decomposition can now be interpreted as a planar spinal open book (see Definition 2.1), and the aforementioned classification result can be recovered using the notion of positive allowable monodromy factorizations. We discuss this example in detail in §2.4.1 and §4.1.

The primary goal of this article is to explain how symplectic fillings of contact 3-manifolds supported by **planar uniform spinal open books** can be classified using monodromy factorizations. This technique is applicable to a much larger class of contact manifolds than those supported by planar open books. The main results are the following theorems from [LHMW18, LHMW20, MRW25].

Theorem A. *Let (M, ξ) be a contact 3-manifold supported by a planar uniform spinal open book and (W, ω) a minimal strong symplectic filling of (M, ξ). Then, (W, ω) is symplectic deformation to a **positive allowable nearly Lefschetz fibration**, i.e., the complement of a neighborhood of positive multisections in a bordered Lefschetz fibration.*

Theorem A can be interpreted in terms of monodromy factorizations as follows.

Theorem B. *Let (M, ξ) be a contact 3-manifold supported by a planar spinal open book π and $\mathcal{B}$ a collection of surfaces that π is uniform with respect to. Then a minimal strong filling of (M, ξ) corresponds to a positive admissible factorization of the monodromy of the spinal open book with respect to some surface $B \in \mathcal{B}$.*

In §4, we outline a strategy for the classification of symplectic fillings based on Theorem B and provide several examples.

Remark 1.9. Note that Theorem B does not claim a one-to-one correspondence. There may be redundancies in the correspondence, and we will see such examples in §4.

In upcoming work [BRW], it is shown that a positive admissible nearly Lefschetz fibration supports a Stein structure, generalizing the result on positive allowable Lefschetz fibrations (PALFs) by [AO02, LHMW18][1]. In particular, all minimal (weak, strong, exact) symplectic fillings of planar uniform spinal open books are symplectically deformation equivalent to Stein fillings. Details about the definition of nearly Lefschetz fibrations, and an associated Stein structure, will be given in §3.

Theorem 1.10. *A positive admissible nearly Lefschetz fibration (PANLF) supports a canonical Stein structure.*

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[1] The "positive" part of the definition can be omitted, since all Lefschetz singularities are by definition positive. We keep it here as a historical continuation of terminology

We will define the term **uniform** condition in §2, but for now, one can understand that this is some natural (technical) condition imposed on a planar spinal open book bounding a nearly Lefschetz fibration. The precise notion of a **positive admissible factorization** is clarified in Definition 2.10.

1.2. **Organization.** In §2 we give definitions pertaining to spinal open books, and also discuss examples and methods of obtaining spinal open book decompositions. In §3 we define and discuss nearly Lefschetz fibrations, and describe associated Stein structures. Then in §4 we show how Theorems A and B can be used to classify symplectic fillings for some families of 3-manifolds. We conclude with some questions for future research in §5.

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2. SPINAL OPEN BOOKS

In this section, we review basic definitions and tools used to study spinal open book decompositions.

2.1. **Spinal open book decompositions.** In this subsection, we review the definition of spinal open book decompositions, which first appeared in [LHMW18, LHMW20]. We then recall the contact structures they support, and characterize the spinal open books that bound (nearly) Lefschetz fibrations.

Definition 2.1. A **spinal open book decomposition** of a closed oriented 3-manifold M is a decomposition $M = M_\Sigma \cup M_P$ (called the **spine** and **paper**, respectively) together with a pair of fibrations $\pi = (\pi_\Sigma, \pi_P)$ such that

$$\pi_\Sigma : M_\Sigma \to \Sigma$$
$$\pi_P : M_P \to S^1$$

where

(1) Σ is a compact oriented surface whose connected components (called **vertebrae**) have nonempty boundary and π_Σ is a trivial fibration with S^1 fiber;
(2) The fibers of π_P are compact oriented surfaces whose connected components (called **pages** and denoted by P) have nonempty boundary and intersect transversely to ∂M_P. The intersection of P and M_Σ consists of fibers of π_Σ;
(3) For every component of ∂M_P, there exist local coordinates $(\phi, t, \theta) \in S^1 \times (-1, 0] \times S^1$ on a collar neighborhood such that $\pi_P(\phi, t, \theta) = m\phi$ for some $m \in \mathbb{N}$. On every component of ∂M_Σ, there exist local coordinates $(s, \phi, \theta) \in (-1, 0] \times S^1 \times S^1$ such that $\pi_\Sigma(s, \phi, \theta) = (s, \phi)$. The number m is called the **multiplicity** of π_P at that boundary component of M_P. On the overlap between M_P and M_Σ the 2-torus coordinates (ϕ, θ) agree, while the interval coordinates are related by $s = -t$;

(4) The paper M_P can be identified with a mapping torus

$$M_P = \mathbb{R} \times P \big/ \sim, \quad (\tau, p) \sim (\tau - 1, \phi(p))$$

and $\pi_P : M_P \to S^1$ is given by $[(\tau, p)] \to [\tau]$. Here, $\phi \in \mathrm{FMod}(P)$ is the monodromy in the framed mapping class group of P. See §2.2 for the definition.

Note that when all vertebrae are disks and the multiplicity of each component of ∂M_P is 1, the definition of spinal open books recovers the definition of ordinary open books. A contact form λ on a spinal open book is called a **Giroux form** if $d\lambda$ is positive on the interior of each page and the Reeb vector field associated to λ is positively tangent to every fiber of π_Σ. We say that a contact structure ξ on M is **supported by** π if ξ is isotopic to a contact structure that admits a Giroux form for π.

Remark 2.2. We will frequently use the 3-tuple (P, ϕ, Σ) as a shorthand for a spinal open book where P, ϕ, and Σ denote the page, monodromy, and vertebra, respectively. Note that this tuple does not uniquely determine the spinal open book, as it omits the specific correspondence between vertebrae and the boundary components of the page.

Definition 2.3. A spinal open book $\pi = (\pi_\Sigma, \pi_P)$ of M is **symmetric** if

(1) all pages are diffeomorphic,
(2) for each vertebra Σ_i ($\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_r$), there exists $k_i \in \mathbb{N}$ such that every page has exactly k_i boundary components in ${\pi_\Sigma}^{-1}(\partial \Sigma_i)$.

We also say that π is **uniform** if it is symmetric and there exists a compact oriented surface B such that

(1) connected components of ∂B are in bijection with the components of M_P;
(2) for each vertebra Σ_i ($\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_r$), there exists a k_i-fold branched cover $\pi_i : \Sigma_i \to B$ such that for each $\gamma \subset \partial \Sigma_i$, $\pi_i|_\gamma$ is an m_γ-fold cover where m_γ denotes the multiplicity of π_P at $\pi_\Sigma^{-1}(\gamma) \subset M_P$.

In the above case we further say that π is **uniform with respect to** B, and we will refer to B as the **base** of the spinal open book. We say that π is **Lefschetz-amenable** if it is uniform and all branched covers have no branch points. These definitions are motivated by the fact that a bordered Lefschetz fibration induces a Lefschetz-amenable spinal open book on its boundary, while a nearly Lefschetz fibration induces a uniform spinal open book. See §3 for the definition of nearly Lefschetz fibrations and see §4 for examples of the construction of a nearly Lefschetz fibration from a uniform spinal open book.

The Lefschetz amenable case has been extensively studied in [LHMW18, LHMW20], while [MRW25] studies the non-Lefschetz-amenable case which necessarily involves the exotic fibers. See §3.2 for more details on exotic fibers. In the non-amenable case, we will refer to a vertebra (and a spine) as **non-trivial** if the map to B has branch points.

We present some examples. For a Lefschetz-amenable spinal open book, see §2.4.1 for a spinal open book for 3-torus $\mathbb{T}^3$. The next example describes a non-uniform spinal open book decomposition.

Example 2.4. Let $P = P_1 \sqcup P_2$ be two sheets of pair of pants and $\Sigma = \Sigma_1 \sqcup \Sigma_2 \sqcup \Sigma_3$ where Σ_1 and Σ_2 are disks and Σ_3 is an annulus. The monodromy of the pages is $\phi = (\phi_1, \phi_2)$, where ϕ_1 interchanges two inner boundary components of P_1 and ϕ_2 interchanges two inner boundary components of P_2. See §2.2 for the definition of a boundary interchange map. To determine the spinal open book, we specify the correspondence between vertebrae and boundary components of the pages. First, Σ_1 and Σ_2 correspond to the outer boundary components of P_1 and P_2, repectively, and Σ_3 correspond to the inner boundary components of both P_1 and P_2. Here, the base B should have two boundary components as there are two connected components in P. However, the disks Σ_1 and Σ_2 cannot be a (branched) cover of B, so the spinal open book cannot be uniform.

2.2. **Framed mapping class group.**

To describe the monodromy of a spinal open book decomposition, we use the **framed mapping class group** of the page. In this section, we review relevant material on framed mapping class groups. For more details, readers are referred to [BH16a].

Definition 2.5. Let M be a manifold. Suppose $P \subset M$ and $U_1, \dots, U_k \subset TM$. The **framed mapping class group** of M relative to P and $U_1, \dots, U_k$ is defined as

$$\begin{aligned}\mathrm{Mod}_P(\Sigma; U_1, \dots, U_k) &:= \pi_0(\mathrm{Diff}^+_P(\Sigma; U_1, \dots, U_k))\\ &:= \pi_0\left(\{\phi \in \mathrm{Diff}^+(\Sigma) \mid \phi|_P = \mathrm{id},\ d\phi(U_i) = U_i\}\right).\end{aligned}$$

For a compact oriented surface Σ, we use the following version of framed mapping class group:

Definition 2.6. Let Σ be a compact oriented surface with n boundary components and $\{p_1, ..., p_n\}$ a set of points on each boundary component. In this case, we define its framed mapping class group as

$$\mathrm{FMod}(\Sigma) := \mathrm{Mod}_{\emptyset}(\Sigma; \{p_1, \dots, p_n\})$$

Here, we identify each p_i with the zero vector in $T_{p_i}\Sigma$.

Remark 2.7. In the previous paper [MRW25], we defined the **spinal mapping class group** $\mathrm{SMod}(\Sigma)$, which turned out to be identical to the framed mapping class group $\mathrm{FMod}(\Sigma)$. Consequently, we use the term "framed mapping class group", as it is a term already used in the literature.

We now describe a set of generators for the framed mapping class group $\mathrm{FMod}(\Sigma)$. In particular, we define the **boundary interchange map** τ'_γ between two boundary components about an arc γ to be as in Figure 4. This map rotates and switches the two boundary components around a small neighborhood of the arc in a counterclockwise direction. For our purpose, however, we instead use a **negatively twisted boundary interchange** $\tau_\gamma = T_{c_1}^{-1} \circ T_{c_2}^{-1} \circ \tau'_\gamma$ that interchanges two boundary components and then negatively twists the boundary components to account for trivializations coming from spinal open book decompositions. See Figure 5.

From now on, we refer to the “boundary interchange map” as the “negatively twisted boundary interchange map” (unless stated otherwise), as it is a base element for monodromy factorizations for spinal open book decompositions.

Theorem 2.8. FMod(Σ) *is generated by Dehn twists about homologically essential simple closed curves on* Σ *and boundary interchange maps between two boundary components about an arc connecting two points on each boundary components.*

2.3. **Monodromy factorizations.** Here, we first introduce a monodromy factorization of a framed mapping class, and then explain how to apply it to a spinal open book setup. To do so, we recall from §2.2 that a boundary interchange map τ_γ interchanges two boundary components of P along an arc γ, as shown in Figure 5.

Let P be a compact oriented surface with boundary, $\phi \in \mathrm{FMod}(P)$, and $\rho\colon G \to \mathrm{Mod}(P)$ a monodromy representation from a group G.

Definition 2.9. Let P, ϕ, and ρ be as above. We further suppose $g \in G$ and $n \in \mathbb{N}$. A **positive admissible factorization of ϕ with respect to** (ρ, g, n) is an expression of $\phi \circ \rho(g)^{-1}$ as a product of:

- n boundary interchange maps between pairs of boundary components of P along arcs that join the boundary components, and
- positive Dehn twists about homologically essential curves on P.

See Figure 14 for an example of such a factorization.

Suppose we have a spinal open book (π_Σ, π_P) of a closed 3-manifold M with a page $P = (P_1, \dots, P_m)$, monodromy $\phi = (\phi_1, \dots, \phi_m)$ where $\phi_i \in \mathrm{FMod}(P)$, and vertebrae $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_r$. Let B be the base of the spinal open book, a compact oriented surface with boundary such that for each $k = 1, \dots, r$, there is a (possibly branched) covering map $\pi_k : \Sigma_k \to B$ with n_k simple branch points. Also suppose (π_Σ, π_P) is uniform with respect to B. Then all pages are diffeomorphic and there is a one-to-one correspondence between each paper component and boundary component of B. To be more specific, for each boundary component ∂_i of B, there exists a corresponding paper component with monodromy ϕ_i that intersects the boundary component $\pi_k^{-1}(\partial_i)$ of Σ_k for every $k = 1, \dots, r$.

Now we consider the following subsets of ∂P:

$$\partial_k P_i := \{c \subset \partial P_i \mid c \text{ intersects a spine component } \Sigma_k \times S^1\}.$$

From now on, we will slightly abuse notation. We first set P_0 as a “model surface” such that there is a diffeomorphism $f_i\colon P_0 \to P_i$ for each $i = 1, \dots, m$. Then we can identify all pages $P_1, \dots, P_m$ with the single surface P_0 and use the monodromy $f_i^{-1} \circ \phi_i \circ f_i$ instead of ϕ_i. We relabel $f_i^{-1} \circ \phi_i \circ f_i$ as ϕ_i to shorten the notation. Also, we fix a monodromy representation $\rho\colon \pi_1(B) \to \mathrm{Mod}(P_0^c)$ where P_0^c is the surface obtained by capping off the boundary components of P_0 that correspond to a non-trivial vertebra. Recall that for each paper component with monodromy ϕ_i, there is a corresponding boundary component ∂_i of B.

Definition 2.10. Let P, ϕ, B, ρ, and n_k be as above. A **positive admissible factorization of ϕ with respect to** (B, ρ) is an expression of each $\phi_i \circ \rho(\partial_i)^{-1}$ for $i = 1, \ldots, m$ as a product of

- n_k boundary interchanges between pairs of boundary components in $\partial_k P_i$ along arcs that join the boundary components for $k = 1, \ldots, r$, and
- positive Dehn twists about homologically essential curves in P_i.

The main purpose of this factorization is to construct a PANLF over B with fiber P_0. We first construct a P_0^c-fiber bundle over B using the monodromy representation ρ. We then introduce exotic fibers by removing positive multisections corresponding to Σ_k with $n_k > 0$, which modifies the fiber P_0^c into P_0. Lastly, add singular fibers by attaching Weinstein 2-handles along the curves associated with the Dehn twists. Note that each connected component of paper M_P, a P_i-bundle over S^1 with monodromy ϕ_i, corresponds to a subbundle of this PANLF restricted to the boundary component ∂_i of B. The total number of exotic fibers is $n_B = n_1 + \cdots + n_r$, and the total number of singular fibers is equal to the total number of Dehn twists across $P_1, \ldots, P_m$.

Now we collect all such factorizations that range over all monodromy representations.

Definition 2.11. The collection of positive admissible factorizations of ϕ with respect to (B, ρ), as ρ ranges over all possible monodromy representations, will be called the **set of positive admissible factorizations of ϕ with respect to** B.

Example 2.12. Figure 13 represents spinal open book decompositions for elliptic torus bundles. However, the factorizations of the monodromies ϕ_1, ϕ_2, and ϕ_3 are not positively admissible, as they contain negative Dehn twists along closed curves. Figure 14 represents the same elliptic torus bundles, but their monodromy factorizations are positively admissible.

We now describe equivalence between two factorizations that induce an equivalent PANLF. The following definition is due to Baykur and Hayano [BH16b].

Definition 2.13 (Generalized Hurwitz equivalence)**.** Let F be a compact oriented surface with boundary. Two positive allowable monodromy factorizations of $\phi \in \mathrm{FMod}(F)$ will be said to be **Hurwitz equivalent** if they are related by the following moves:

(1) **Elementary transformation**, which changes a factorization as follows:

$$\eta_{k+l} \cdots \eta_{i+1}\eta_i \cdots \eta_1 \longleftrightarrow \eta_{k+l} \cdots (\eta_{i+1}\eta_i\eta_{i+1}^{-1})\eta_{i+1} \cdots \eta_1$$

(2) **Global conjugation**, which changes each member of a factorization by the conjugation of some mapping class $\psi \in \mathrm{Mod}_\partial(F)$:

$$\eta_{k+l} \cdots \eta_1 \longleftrightarrow (\psi\eta_{k+l}\psi^{-1}) \cdots (\psi\eta_1\psi^{-1})$$

(3) **Framing conjugation**, which changes a factorization as follows:

$$\eta_{k+l} \cdots \eta_{i+1}\eta_i\eta_{i-1} \cdots \eta_1 \longleftrightarrow \eta_{k+l} \cdots \eta_{i+1}(T_\delta\eta_i T_\delta^{-1})\eta_{i-1} \cdots \eta_1$$

where δ is a simple closed curve parallel to a boundary component of F.

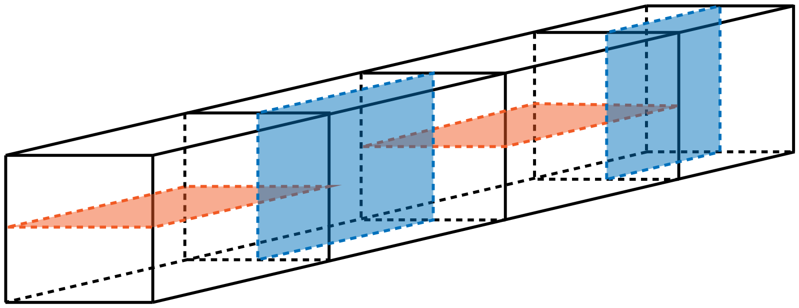

FIGURE 1. The 3-torus $\mathbb{T}^3$. The top and bottom faces are identified, as are the front and back, and the left and right. The blue annuli represent the pages of the spinal open book, and the red annuli represent the vertebrae.

The generalized Hurwitz moves naturally arise as a consequence of choices made when obtaining a factorization from a PANLF as we will see in §3. As a result, we obtain the following result:

Proposition 2.14. *Two Hurwitz equivalent monodromy factorizations give rise to two symplectic PANLFs that are deformation equivalent.*

2.4. **Construction of spinal open book decompositions.**

In this section we discuss some methods to construct supporting spinal open book decompositions for contact 3-manifolds.

2.4.1. *The Stein fillable contact structure on the 3-torus.* Eliashberg [Eli96] showed that there exists a unique Stein fillable contact structure ξ_0 on $\mathbb{T}^3$. In [Wen13, LHMW20], a spinal open book decomposition for $(\mathbb{T}^3, \xi_0)$ was introduced, as shown in Figure 1. The first and third $\mathbb{T}^2 \times I$ pieces in the figure are spines and the second and fourth are papers. The blue annuli are pages, and the red annuli are vertebrae. This spinal open book decomposition naturally arises as the boundary of the bordered Lefschetz fibration $A \times A$, with pages $\partial A \times A$ and vertebrae $A \times \partial A$, where $A = S^1 \times I$ is an annulus. The monodromy of the paper is id.

Now we verify that this spinal open book decomposition actually supports ξ_0. According to §2.4.3, a spine with an annulus vertebra is equivalent to a relative open book deposition with the word $(aba)^{-2}$ as shown in the shaded region in Figure 12. Since our spinal open book consists of two annulus pages with identity monodromy, and two annulus vertebrae, it is equivalent to an ordinary open book decomposition with the word $(aba)^{-4}$, which supports ξ_0 by Van Horn-Morris [VHM07].

2.4.2. *Blown up summed open books.* There is a natural operation on a regular contact open book decomposition called **binding sum** that gives rise to a spinal open book. Let $\pi : M \setminus B \to S^1$ be an open book decomposition with at least two binding components $B_1, B_2 \subset B$. Topologically, one removes neighborhoods of B_1 and B_2 and glues the resulting boundary tori via an orientation reversing diffeomorphism which maps the boundaries of the pages to each other and meridians to meridians (with opposite orientation).

This is equivalent to attach two boundary components to each boundary component of $\mathbb{T}^2 \times I$. Thus the resulting manifold naturally has the structure of a spinal open book decomposition with an annulus vertebra replacing the two disk vertebrae, and the supporting contact structure is obtained by a contact fiber sum along B_1 and B_2 of the original open book decomposition.

2.4.3. *Torus bundles and relative open books.* Here, we construct planar spinal open book decompositions for **rotational contact structures** on torus bundles, using the technique of **relative open book decompositions** introduced by Van Horn-Morris [VHM07].

we first review rotational contact structures on torus bundles. Let $A \in SL_2(\mathbb{Z}) = \mathrm{Mod}(\mathbb{T}^2)$. Then we define the torus bundle with monodromy A by

$$T_A := \mathbb{T}^2 \times \mathbb{R}/(x,t) \sim (Ax, t-1).$$

Moreover, T_A is called **elliptic, parabolic**, and **hyperbolic**, respectively, as $|\mathrm{tr}(A)|$ is less than 2, equal to 2, or greater than 2. We consider a **rotational contact structure** ξ_n for an integer $n \geq 0$ defined by

$$\xi_n = \ker[\sin\phi_e(t)\,dx + \cos\phi_n(t)\,dy], \quad (x,y) \in \mathbb{T}^2,\ t \in \mathbb{R}/\mathbb{Z}$$

where $\phi_n \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ satisfies

$$n\pi \leq \sup_{t \in \mathbb{R}/\mathbb{Z}} (\phi_n(t+1) - \phi_n(t)) < (n+1)\pi.$$

We say ξ_n has **$n\pi$-twisting**. Each torus bundle only admits either odd or even rotational contact structures. If A is parabolic, then A is conjugate to

$$\pm A_k = \pm \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

for some $k \in \mathbb{Z}$. We say a parabolic torus bundle T_A is **positive** if A is conjugate to A_k and **negative** if A is conjugate to $-A_k$. Let $T_+(k) := T_{A_k}$ and $T_-(k) := T_{-A_k}$. Note that $T_+(k)$ only admits even rotational contact structures ξ_{2n} for $n \geq 1$ and $T_-(k)$ only admits odd rotational contact structures ξ_{2n+1} for $n \geq 0$.

Next, we review relative open book decompositions, introduced by Van Horn-Morris [VHM07]. Let P be a compact oriented surface with boundary and $\phi \in \mathrm{Mod}_\partial(P)$. Consider the mapping torus

$$P_\phi = P \times [0,1]/\sim$$

where $(p,1) \sim (\phi(p),0)$ for $p \in P$. Now pick a partition of ∂P into two sets ∂_B and ∂_T. Each component of ∂_B is called **binding circles**, and each component of ∂_T is called **boundary circles**. We foliate the boundary tori $\partial_B \times S^1 \subset \partial P_\phi$ into $\{x\} \times S^1$, where $x \in \partial_B$. From this, we can construct a 3-manifold M with torus boundary components by collapsing the foliating circles in each component of $\partial_B \times S^1$. We say $(P, \phi, (\partial_B, \partial_T))$ is a **relative open book decomposition** of M. If $\partial_T = \emptyset$, we obtain a closed manifold with a regular open book decomposition.

As in the case of regular open book decompositions, there is a contact structure compatible with a given relative open book decomposition.

Definition 2.15. Let M be a compact oriented 3-manifold and $(P, \phi, (\partial_B, \partial_T))$ a relative open book of M. A contact structure ξ on M is said to be **compatible with** $(P, \phi, (\partial_B, \partial_T))$ if there exists a contact form α where $d\alpha$ restricts to a symplectic form on each page, and the foliation given by ξ on ∂M agrees with the foliation given by boundary circles in ∂_T.

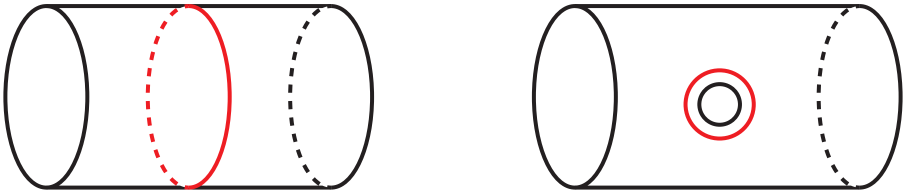

FIGURE 2. Relative open book decompositions for a and b^{-1}. For a^{-1} and b we use the negative Dehn twists instead.

We now review a method for representing relative open books of $S^1 \times I \times S^1$ using **words**. Four relative open books, which correspond to the set of letters $\{a, a^{-1}, b, b^{-1}\}$ as depicted in Figure 2, act as building blocks for a relative open book decomposition supporting a contact structure on $\mathbb{T}^2 \times I$, or a torus bundle over S^1. A torus bundle obtained by gluing these pieces is denoted by the corresponding word in terms of a, a^{-1}, b and b^{-1}. For example, the left drawing of Figure 12 represents an open book decomposition corresponding to the word $a^k(aba)^{-4}$.

Lastly, we find a relative open book of a spine component with annular vertebra that agrees on the boundary of the spine component with the page foliation of the spinal open book, which is a key to construct planar spinal open book decompositions for rotational torus bundles. Note that a spine component with an annulus vertebra is also homeomorphic to $S^1 \times I \times S^1$.

Proposition 2.16 (Min-Roy-Wang [MRW25])**.** *The relative open book for a spine component of an annular vertebra is contactomorphic to a relative open book with the word* $(aba)^{-2}$.

Using this proposition, we can convert any genus-1 open book decomposition containing a word $(aba)^{-2}$ into a planar spinal open book decompositions as follows:

(i) Consider a genus-1 open book decomposition that contains the word $(aba)^{-2}$.
(ii) Remove the relative open book corresponding to $(aba)^{-2}$ from the entire open book decomposition. Then the remaining part become a page of the spinal open book, and the relative open book becomes an annular vertebra.
(iii) Repeat (ii) on the page of the spinal open book and we obtain multiple annular vertebrae and page components. This new spinal open book supports the same contact structure as the original open book.

See Figure 12 for an example. An open book decomposition with the word $(aba)^{-4}a^k$ supports a positive parabolic torus bundle $(T_+(k), \xi_2)$. Now we remove two relative open books corresponding to $(aba)^{-2}$ and replace them with annular vertebrae. The remaning part becomes two annular pages with monodromies a^k and id, respectively.

Exercise 2.17. Find planar spinal open book decompositions for π-twisting elliptic torus bundles.
(*Hint: Find the words for the bundles from* [VHM07, Theorem 4.3.1].)

Exercise 2.18. Show that the 2π-twisting 3-torus has the word $(aba)^{-4}$.
(Hint: Convert a spinal open book for the 3*-torus from §2.4.1 into an honest open book.)*

Exercise 2.19. Show that any rotational torus bundle with at least π-twisting admits a planar spinal open book decomposition.
(Hint: Use Exercise 2.18 to show that these bundles admit a genus-1 open book containing the word $(aba)^{-2}$.)

3. NEARLY LEFSCHETZ FIBRATIONS

In this section we describe the topological structure on a symplectic 4-manifold with boundary that induces a spinal open book on the boundary. Just as bordered Lefschetz fibrations over the disk induce open book decompositions on the boundary, these **nearly Lefschetz fibrations** induce spinal open book decompositions of the boundary. We describe the notion of a bordered Lefschetz fibration over a surface with boundary in § 3.1. These give rise to "amenable" spinal open books on the boundary. Then, in § 3.2, we describe the "singularity-at-infinity" that arises when studying the foliations coming from planar spinal open books, called **exotic fibers**. We give an explicit local model and study the induced monodromy on an S^1-bundle over a curve in the base enclosing such a these singularity. Then in § 3.3, we describe nearly Lefschetz fibrations, and how to associate Stein structures to them.

3.1. **Bordered Lefschetz fibrations.**

A spinal open book decomposition naturally arises as the boundary of a bordered Lefschetz fibration. In this section we recall the definition of a bordered Lefschetz fibration by following the conventions of [LHMW18]. Let E be a compact oriented connected 4-manifold with corners. We decompose the boundary of E into two parts along the corners:

$$\partial E = \partial_h E \cup \partial_v E.$$

We further assume that the intersection of $\partial_h E$ and $\partial_v E$ is a collection of 2-tori.

Unlike [LHMW18], since we will not be carrying out any analysis near the corners, we will treat E as a smooth manifold, assuming we already rounded the corners. Let B denote a compact oriented connected surface with connected boundary.

Definition 3.1. A **bordered Lefschetz fibration** of E over B is a smooth map $\Pi : E \to B$ with finitely many interior critical points $E^{\mathrm{crit}} \subset E^\circ$ and critical values $B^{\mathrm{crit}} \subset B^\circ$ such that

1. $\Pi^{-1}(\partial B) = \partial_v E$ and $\Pi|_{\partial_v E} : \partial_v E \to \partial B$ is a smooth fiber bundle.
2. $\Pi|_{\partial_h E} : \partial_h E \to B$ is a smooth fiber bundle.
3. $(\Pi|_{\partial_h E}, \Pi|_{\partial_v E})$ is a spinal open book decomposition of ∂E.
4. For each $p \in E^{\mathrm{crit}}$ and $\Pi(p) \in B^{\mathrm{crit}}$, there are local holomorphic coordinates on E centered at p and on B centered at $\Pi(p)$ such that $\Pi(z_1, z_2) = z_1^2 + z_2^2$.

(5) For each $z \in B$, the fiber $E_z := \Pi^{-1}(z)$ is connected and has nonempty boundary in $\partial_h E$.

The fibers $\Pi^{-1}(z)$ for $z \in B \setminus B^{\text{crit}}$ are called **regular fibers**, and for $z \in B^{\text{crit}}$ are called **singular fibers**. The regular fibers are all homeomorphic smooth compact oriented surfaces with boundary. The singular fibers are smoothly immersed connected surfaces with positive transverse self-intersections – topologically they are obtained by taking a regular fiber and pinching a curve on it down to a point; the curve that gets pinched is called a **vanishing cycle**. A Lefschetz fibration Π is **allowable** if all the vanishing cycles are homologically essential curves on a regular fiber.

It is well-known that allowable bordered Lefschetz fibrations admit a Stein structure.

Theorem 3.2 ([BVHMLW15, Theorem 3.9][LHMW18, Theorem B])**.** *Let E be a 4-dimensional allowable bordered Lefschetz fibration and π is an induced spinal open book of ∂E. Then, E admits a (well-defined up to Stein homotopy) Stein structure which is a Stein filling of the contact manifold $(\partial E, \pi)$. Moreover, given two bordered Lefschetz fibrations E_1 and E_2 that fill $(\partial E, \pi)$, their Stein structures can be chosen to induce the same contact structure on the boundary.*

3.2. **Exotic fibers.** In this section, we will recall the local model of a neighborhood of an exotic fiber and explain how to see the local monodromy in terms of a boundary interchange in Lemma 3.6. This should be reminiscent of the story Lefschetz singularities, where we have holomorphic local models $(z_1, z_2) \mapsto z_1^2 + z_2^2$ and the local monodromy is a right-handed Dehn twist around an homologically essential curve. See [GS99]. However, it should be emphasized early on that, as opposed to the Lefschetz singular fibers, the exotic fibers exhibit a "singularity-at-infinity" phenomenon, incorporating a mapping class that interchanges boundary components.

The original definition of exotic fibers is a technical one: they are the main level of buildings which appear in the compactification of certain moduli space of curves that [LHMW20] analyzes. Since this lecture note focuses on the application of spinal open books and symplectic fillings potentially containing exotic fibers, we will focus on the topological description and refer the readers to [MRW25, §4] for a careful treatment of the local model of the exotic fibers in terms of SFT moduli spaces and the proof of uniqueness of the local model.

Definition 3.3 (Local model)**.** Consider the sequence of curves

$$u_c : \mathbb{C}\backslash\{\pm 1\} \to (\mathbb{C}\backslash\{0\}) \times \mathbb{C},$$

defined by

$$u_c(z) = (c(z^2 - 1), \sqrt{c}z). \tag{1}$$

Consider the projection map

$$\Pi_0 : (\mathbb{C}\backslash\{0\}) \times \mathbb{C} \to \mathbb{C} \text{ sending } (z_1, z_2) \mapsto z_2^2 - z_1, \tag{2}$$

whose level sets are identified with the image of u_c.

Remark 3.4. The moduli space of curves $u_c(z)$ in Equation (1) is parameterized by the domain of $c \in \mathbb{C} \setminus \{0\}$. The reparameterization by $z' := \sqrt{c}z$ allows us to compactify the moduli space at $c = 0$ with a limiting building as in Figure 6.

Remark 3.5. Definition 1 as well as the understanding of pseudoholomorphic foliations in [LHMW20] are both in the non-compact setting in the completion of the minimal strong symplectic fillings. As is often useful to talk about symplectic fillings as a compact manifold with contact boundary, the corresponding local model also needs to be compactified to be technically correct. We define the **compactified local model** to be the restriction of the above model to $(\mathbb{C} \setminus B) \times \mathbb{C}$, which corresponds to cutting off the cylindrical ends at the punctures. Here B is defined to be the closed ϵ-disk about the origin in $\mathbb{C}$. The restricted foliation on $(\mathbb{C} \setminus B) \times \mathbb{C}$ is as follows. For u_c, $\mathrm{im}(u_c) \cap (\mathbb{C} \setminus B) \times \mathbb{C} = \{u_c(z) | |z^2 - 1| > \frac{\epsilon}{|c|}\}$. Thus, for $|c| > \epsilon$, $\mathrm{im}(u_c) \cap (\mathbb{C} \setminus B) \times \mathbb{C}$ is topologically a pair of pants, while for $|c| < \epsilon$ $\mathrm{im}(u_c) \cap (\mathbb{C} \setminus B) \times \mathbb{C}$ is topologically an annulus, with the curves corresponding to $|c| = \epsilon$ having a singularity. So in the compactified local model, we get **regular curves**, corresponding to $\mathrm{im}(u_c)$ for $c > \epsilon$, and a $\mathbb{D}^2$-worth of **exotic curves**, corresponding to $\mathrm{im}(u_c)$ for $c \leq \epsilon$, which we call an **exotic neighborhood**.

We will see in Lemma 3.6 that this gives a model that interchanges the two punctures as one goes around $0 \in \mathbb{C}$ as in Figure 5. Further, this local model of the exotic fiber also allows us to show that the number of exotic points is equal to the number of the branch points of $\Pi|_\Sigma$ in Theorem C. To describe the monodromy of the fibers in a compactified filling around the exotic neighborhood, we will talk about **boundary components** of the fibers in the compactified model, which correspond to punctures in the local model.

Lemma 3.6. *Consider the fibers in the compactified local model. Outside the exotic neighborhood, the fibers are pairs of pants. The monodromy of the bundle with pairs of pants fibers over S^1, around the exotic fiber u_0, is given by interchanging two boundary components counter-clockwise along an arc and simultaneously rotating the boundary components by π clockwise.*

Proof. In the model from Definition 3.3, consider the following disk neighborhood V of $0 \in \mathbb{C}$. Let $\delta > 2\epsilon$ be large enough so that the δ-disk neighborhood $V \subset \mathbb{C}$ contains the image of the exotic neighborhood. We parameterize V so that $\Pi^{-1}(0)$ corresponds to the exotic fiber. Let $\zeta := \partial V$. Over each point $x \in \zeta$, $\Pi^{-1}(x)$ is a pair of pants. We will characterize the monodromy of this bundle with pair of pants fibers around ζ. Also consider a chart U of the filling W around the exotic neighborhood, as above. By construction, topologically $\Pi^{-1}(V) - U$ is a trivial $\Sigma - P$ bundle over the disk V, where P is a pair of pants. In other words, the monodromy is supported in P.

Now we parameterize $\zeta = \{\delta e^{2\pi i t} | t \in [0, 1]\}$ and understand the monodromy around it. First, fix a trivialization on the fibers $\Pi^{-1}(\delta e^{2\pi i t})$, such that the bundle over ζ is trivial outside P. We can parameterize the fibers in $\Pi^{-1}(\zeta) \cap U$ as in Definition 3.3, i.e., the trivialization is given by

$$z \mapsto (e^{2\pi i t}(z^2 - 1), e^{\pi i t} z) \tag{3}$$

for $z \in \Pi^{-1}(\delta) \cap U$. Projecting to the second coordinate, the trivialization we need to consider is H_t such that

$$H_t : z \mapsto e^{\pi i t} z.$$

We can parameterize P as $\{|z|^2 \leq 2\} \cap \Pi^{-1}(\delta) \cap U$, so outside that we need to modify the above trivialization so that the bundle over ζ is trivial. Consider the function f as shown in Figure 3, and consider the trivialization

$$h_t : z \mapsto e^{f(|z|)\pi i t} z. \tag{4}$$

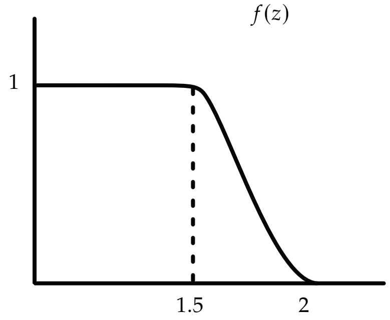

FIGURE 3. The bump function needed to trivialize the monodromy for large $|z|$.

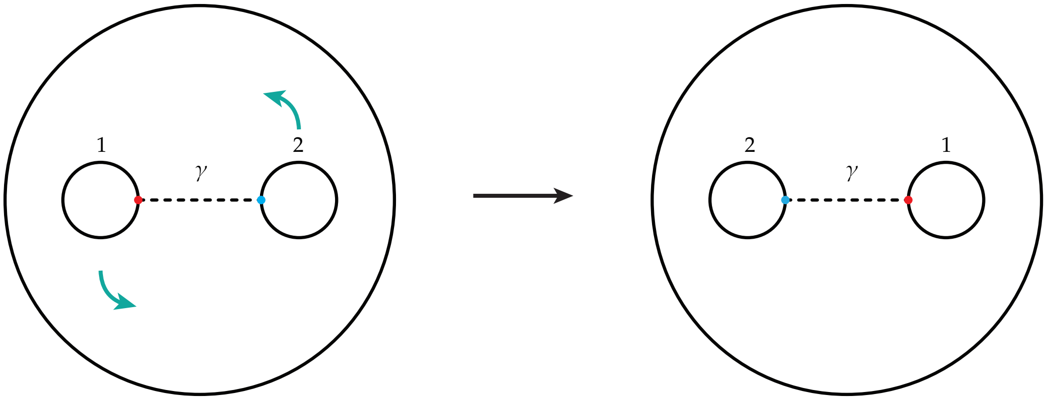

FIGURE 4. Comparing the trivializations h_0 and h_1 after making the bundle trivial away from the pair of pants neighborhood.

Now, compare the trivializations h_1 and h_0. This is shown in Figure 4 for the compactified model.

The above modification of the trivializations ensures that we have localized the monodromy into the pair of pants neighborhood on a fiber. But we are not done yet, since to understand the monodromy of the bundle, in the sense of (3) and (4) in Definition 2.1, we further need to trivialize the bundle near the boundaries of the fibers, to agree with the model of π_P near this component of $\partial M_P \subset \Pi^{-1}(\zeta)$. Observe that the component of ∂M_P in this local model has multiplicity 2, again in the sense of Definition 2.1. Let us refer to the component of ∂M_P as T, and give it coordinates (ϕ, θ), where θ is the S^1-coordinate for the boundary component on a page. Locally π_P has the form $\pi_P|_T(\phi, \theta) = 2\phi$. However, the trivializations in Equation 4 do not agree on T, for different fibers and their intersections with T. To make them agree, a clockwise 2π-rotation of each of the boundary components of P needs to happen. Using a bump function as before, which is 0 on the boundary components and 1 just outside, the trivialization h_1 looks like as in Figure 5. By an isotopy, we can see that the monodromy is exactly a positive half-twist and a half clockwise twist of each of the boundary components, as described in Figure 5. ◻

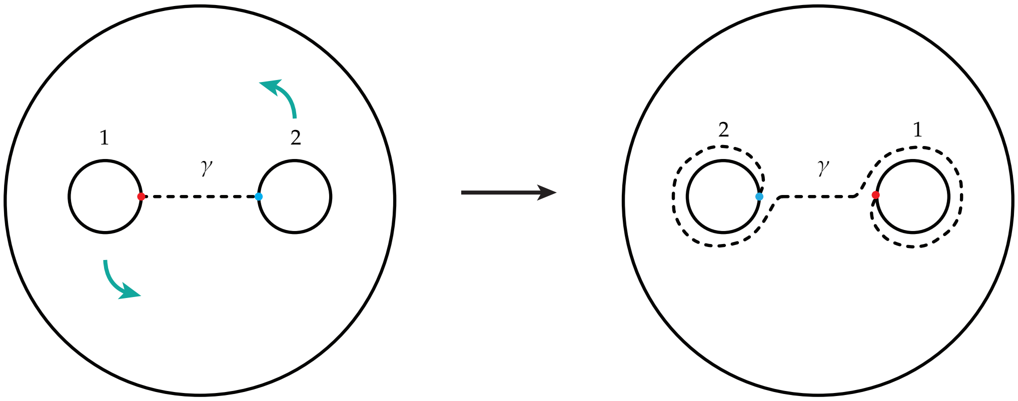

FIGURE 5. A local description of a **boundary interchange** τ_γ, where γ is the equatorial arc connecting the two boundary components in the interior.

In [MRW25], we show that this local model is unique up to symplectic deformation. This involves understanding the asymptotic neighborhood of finite-energy pseudoholomorphic curves asymptotic to a doubly-covered elliptic orbit. Theorem A is then a direct result of this local model, together with the curve classification in [LHMW20, Proposition 1.30] in terms of regular, singular, and exotic fibers. In addition, the local model gives us the following.

Theorem C. *Let (W, ω) be a minimal strong filling of a contact manifold (M, ξ) supported by a planar spinal open book. Then, the number of exotic fibers is equal to the number of branch points of $\Pi|_{S_i}$, where Π is the same Lefschetz fibration as in Theorem A. Furthermore, the monodromy around an exotic fiber in a compactified filling is a boundary interchange on the nearby regular fibers, given by Figure 5.*

This implies that (W, ω) has the structure of a nearly Lefschetz fibration, whose definition we will see now in the following subsection.

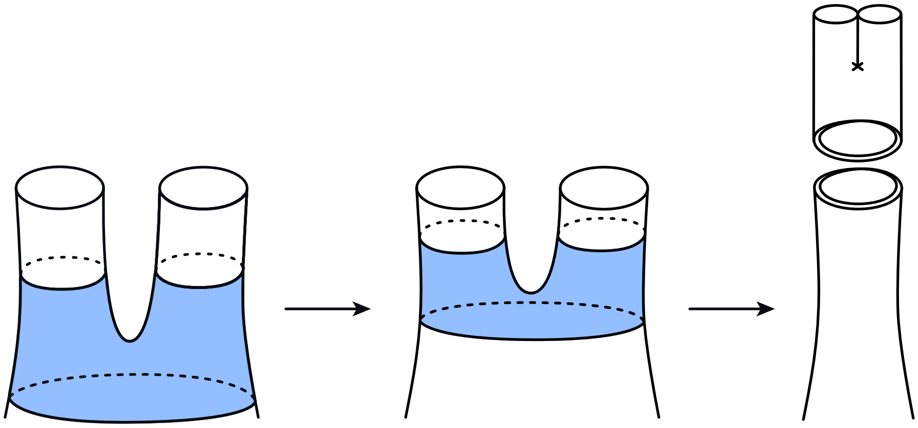

FIGURE 6. A local description of the SFT limit from regular fibers to an exotic fiber.

3.3. PANLF and Stein structures.

3.3. PANLF and Stein structures. For a symplectic filling of an honest planar open book decomposition with suitable conditions, a pseudoholomorphic foliation gives a Lefschetz fibration [Wen10]. However, symplectic fillings of a general spinal open book admit a generalized version of a Lefschetz fibration, which we call a **nearly Lefschetz fibration**. In particular, it gives rise to a uniform spinal open book on its boundary, which is non-Lefschetz-amenable in general.

Definition 3.7. A **nearly Lefschetz fibration** of E over B is a smooth map $\Pi : E \to B$ with finitely many interior critical points $E^{\mathrm{crit}} \subset E^{\circ}$ and critical values $B^{\mathrm{crit}} \subset B^{\circ}$, and finitely many **exotic points** $B^{\mathrm{exot}} \subset B^{\circ}$ such that:

(1) $\Pi^{-1}(\partial B) = \partial_v E$ and $\Pi|_{\partial_v E} : \partial_v E \to \partial B$ is a smooth fiber bundle.
(2) For each $p \in E^{\mathrm{crit}}$ and $\Pi(p) \in B^{\mathrm{crit}}$, there are local holomorphic coordinates on E centered at p and on B centered at $\Pi(p)$ such that $\Pi(z_1, z_2) = z_1^2 + z_2^2$.
(3) All fibers $E_z := \Pi^{-1}(z)$ for $z \in B$ are connected and have nonempty boundary in $\partial_h E$.
(4) For each $c \in B^{\mathrm{exot}}$, there are local holomorphic coordinates $\psi \colon U_c \to \mathbb{C}$, where $U_c \subset B^{\circ}$, sending c to 0 and $\phi \colon V_c \to (\mathbb{C} - D) \times \mathbb{C}$, where $V_c \subset N(\partial_h(E)) \cap \Pi^{-1}(U_c)$ and D is a small neighborhood of $0 \in \mathbb{C}$, such that the following diagram commutes

$$\begin{tikzcd}
V_c \arrow[r, "\phi"] \arrow[d, "\Pi"] & (\mathbb{C} - D) \times \mathbb{C} \arrow[d, "\Pi_c"] \\
U_c \arrow[r, "\psi"] & \mathbb{C}
\end{tikzcd}$$

where $\Pi_c(z_1, z_2) = z_2^2 - z_1$.

(5) The boundary $\partial E = M$ admits a spinal open book decomposition, such that $\partial_v E = M_P$, $\partial_h E = M_\Sigma$, $\pi_P = \Pi|_{\partial_v E}$, and (unions of[2]) fibers of π_Σ are the fibers of $\Pi|_{\partial_h E}$, away from the V_c corresponding to $c \in B^{\text{exot}}$.

In the case of a nearly Lefschetz fibration, we have **regular** and **singular fibers** as in the case of a bordered Lefschetz fibration. The difference lies in the exotic points. We understood the local model of exotic points in the previous subsection. To reiterate, let $c \in B^{\text{exot}}$ and $z \in U_c \subset B^o$, we have that $\Pi^{-1}(z)$ has the topology of a regular fiber near the boundary of U_c, but has one fewer boundary component in a neighborhood of the origin in U_c. We call all the fibers that have one fewer boundary component the **exotic fibers**[3] as explained in Remark 3.5. However, for notational convenience, when we refer to an exotic fiber, we always think of the "genuinely exotic" one at $0 \in \mathbb{C}$.

A nearly Lefschetz fibration Π is **allowable** if all vanishing cycles are homologically essential curves on a regular fiber. In analogy with positive allowable Lefschetz fibrations that are studied in the literature and denoted PALFs, we will also refer to these structures as positive allowable nearly Lefschetz fibrations and abbreviate them to PANLFs.

Remark 3.8. Under the biholomorphism $(z_1, z_2) \mapsto (z_2^2 - z_1, z_2)$ from $\mathbb{C}^2 \setminus \{(z^2, z)\}$ to $\mathbb{C} \setminus \{0\} \times \mathbb{C}$, one can identify the neighborhood of an exotic fiber in the above, with the complement of a branch point of a multisection in a Lefschetz fibration as described in [BH16a, §3].

We now sketch the proof of Theorem 1.10 endowing a PANLF a Stein structure. The full proof can be found in [BRW].

For a bordered Lefschetz fibration, a Weinstein structure can be described by writing down Weinstein models near the regular fibers and the singular fibers, and then patching them up. We write down the details for a manifold W supported by a nearly Lefschetz fibration $\Pi : W \longrightarrow \Sigma$, with a generic fiber S. Both S and Σ are compact oriented surfaces with boundary of genera g_S and g_Σ. Let $\text{Exot}(\Pi) \subset \Sigma$ denote the images of the (isolated) exotic fibers and let $\text{Crit}(\Pi) \subset \Sigma$ denote the set of (isolated) critical values, which are assumed to be disjoint from $\text{Exot}(\Pi)$.

As a preparation step, let's first examine a Weinstein model near exotic fibers, since those near the regular and singular fibers have been done previously, e.g. see [Ozb15]. Given $c \in \mathcal{M}_{\text{exot}} \subset \Sigma$, and a sufficiently small neighbourhood $c \in U_c \subset \Sigma$, the preimage $\Pi^{-1}(U_c)$ contains an open subset $V_c \cong A_\delta \times \mathbb{D}^2$ with complex coordinates (z_1, z_2) so that Π looks like $z_2^2 + z_1$ on V_c[4], with 0 corresponding to c. Here $A_\delta := \{\delta < |z| < 1\}$ denotes

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[2] Recall that by the definition of spinal open books, the fibers of the map π_Σ has to be connected. Therefore, here the disjoint union of the fibers of π_Σ corresponds to (possibly disconnected) fibers of $\Pi|_{\partial_h E}$.

[3] In [LHMW20], the authors discuss exotic fibers in the context of completed fillings, and a finite number of exotic fibers are present in the completed filling. However, since we are looking at compact fillings, cutting off a cylindrical end, we obtain a disk worth of exotic fibers for every single exotic fiber in the completed filling.

[4] In Definition 3.3, the local model is given by $z_2^2 - z_1$. We pick the biholomorphic local model $z_2^2 + z_1$ here simply so that the vector field $\text{Re}(z_1)$ points to the right.

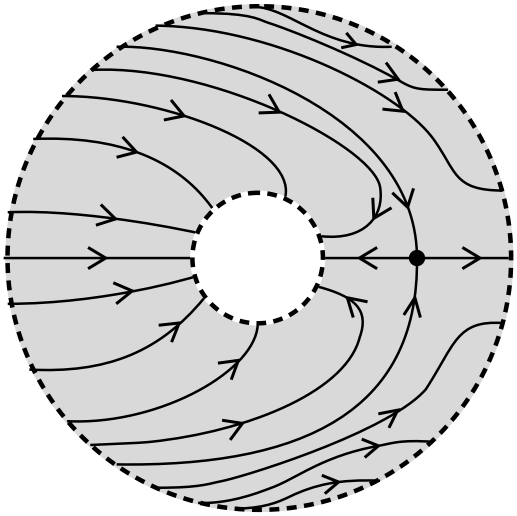

FIGURE 7. Open annulus and the gradient vector field of f_A.

an open annulus around 0 in $\mathbb{C}$ as shown in Figure 7. Let f_A be a function on the annulus whose gradient vector field is shown as in Figure 7. In particular, $f_A = \mathrm{Re}(z_1)$ for $z_1 \in A_\delta$ with norm close to 1. On $A_\delta \times \mathbb{D}^2$, consider the function $f_{\mathrm{exot}} := f_A + x_2^2 - y_2^2$.

We can ensure that f_{exot}, along with the standard complex structure on $A_\delta \times \mathbb{D}^2$ induced from $\mathbb{C} \times \mathbb{C}$, induces a Weinstein structure on $A_\delta \times \mathbb{D}^2$, compatible with the standard symplectic structure, with the Liouville vector field given by ∇f_{exot}.

Now, we are ready to describe a particular Weinstein structure on W as follows (see Figure 8):

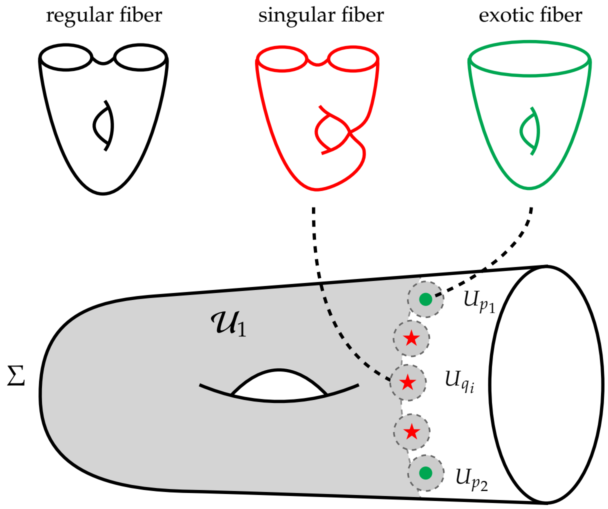

FIGURE 8. Constructing a Stein structure on a PANLF using local models around critical and exotic points.

(1) Consider a Weinstein structure on S given by a function f_S with a single index 0 critical point and $2g_s$ index 1 critical points, and a Weinstein structure on Σ given by a function f_Σ with a single index 0 critical point and $2g_\Sigma$ index 1 critical points.

(2) Pick a finite open cover with open disks $\mathcal{U} = \cup_x U_x$ of Σ such that for every critical value $q \in \mathrm{Crit}(\Pi)$, there is a unique open set U_q containing q. Also, we assume that for any image $p \in \mathrm{Exot}(\Pi)$ of an exotic fiber, there is a unique open set U_p containing p. We ensure that $\cup_{x \notin \mathrm{Crit}(\Pi) \cup \mathrm{Exot}(\Pi)} U_x =: \mathcal{U}_1 \cup \mathcal{U}_2$ is connected, and $\mathcal{U}_1$ is homeomorphic to Σ.

(3) **Regular fibers.** Note that $\Pi^{-1}(\mathcal{U}_1)$ is diffeomorphic to $\Sigma \times S$. Consider the function $f_0 = f_S + f_\Sigma$ to obtain a Weinstein structure on this. Since $\mathcal{U}_1$ is away from the critical points and exotic points, we can ensure that f_0 is arbitrarily close to the real part of Π in some choice of coordinates over the rest $\mathcal{U}_1 \subset \Sigma$. and positively transverse to the boundary of W. We can ensure these choices of coordinates agree with the choices made for the local models near critical and exotic fibers.

(4) **Singular fibers.** Recall that near any critical point c of Π, there are complex local coordinates (z_1, z_2) on a neighbourhood $V_c \subset W$ such that $\Pi|_{V_c} : V_c \to \Sigma$ looks like $z_1^2 + z_2^2$. Consider the function $f_q = x_1^2 + x_2^2 - y_1^2 - y_2^2$ in these coordinates, where $z_j = x_j + iy_j$. Given any critical value q of Π, a critical point c such that $\Pi(c) = q$, and an open set U_q containing q, describe $\Pi^{-1}(U_q) = V_c \cup V_c'$ where V_c' is also open and avoids c. We can choose U_q so that $U_q \subset \Pi(V_c)$, which in particular ensures that V_c' is a trivial $(S - A)$-bundle over the disk U_q, where A is an annulus neighbourhood of the vanishing cycle corresponding to the critical value q. Since $f_q = \mathrm{Re}(\Pi)$ and q is in the interior, we can argue it matches up with f_0 directly on $V_c' \cap \Pi^{-1}(\mathcal{U}_1)$. We thus extend f_0 to a Weinstein structure on $\Pi^{-1}(\mathcal{U}_1)$ to $\Pi^{-1}(\mathcal{U}_1 \cup \bigcup_{q \in \mathrm{Crit}(\Pi)} U_q)$.

(5) **Exotic fibers.** Now, near any $p \in \mathrm{Exot}(\Pi)$, we can again find an open set U_p and observe that $\Pi^{-1}(U_p) = V_d \cup V_d'$, where $d \subset V_d$ is a neighbourhood meeting the boundary where Π takes the form $\Pi|_{V_d} : \mathbb{C} - B \to \mathbb{C}$, $\Pi(z_1, z_2) = z_2^2 + z_1$, with $V_d \cong (\mathbb{C} - B) \times \mathbb{C}$, such that B is a ball around the origin. Similarly as above, V_d' is a trivial $S - P$ bundle over the disk, where P is a pair of pants. On V_d, define the function $f_p = f_A + x_2^2 - y_2^2$ and observe that on $V_d \cap \Pi^{-1}(\mathcal{U}_1)$, the function f_p agrees with $\mathrm{Re}(\Pi)$. Thus we can extend f_0 over $\Pi^{-1}(U_p)$, and thus the Weinstein structure is defined on $\Pi^{-1}(\mathcal{U}_1 \cup \bigcup_{z \in \mathrm{Crit}(\Pi) \cup \mathrm{Exot}(\Pi)} U_z)$.

(6) Once the above steps are done for every critical and exotic point, the Weinstein structure can be extended with no critical points over the trivial cobordism described by $\Pi^{-1}(\mathcal{U}_2)$.

(7) It will be shown in [BRW] that the Weinstein structure above induces a contact form on the boundary which is a Giroux form for the spinal open book induced on the boundary from the PANLF.

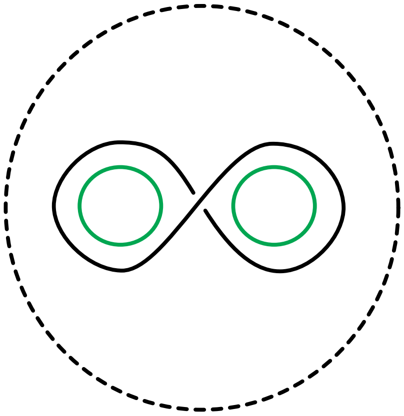

FIGURE 9. The attaching curve for the Weinstein 2-handle cobordism corresponding to an exotic fiber, living in the neighbourhood of a page of the spinal open book.

Exercise 3.9. Show that the boundary interchange mapping class corresponds to a Weinstein cobordism between the relevant contact manifolds, by attaching a Weinstein 2-handle along a Legendrian realization of the knot shown in Figure 9, that lives in a neighbourhood of the pages of the spinal open book. *Hint: Use the local model near the exotic fibers and the definition of the Weinstein function to identify the attaching sphere and the belt-sphere of the handle (up to an isotopy) in neighborhoods of the fibers of* Π*, and conclude that the surgery is smoothly a 0-surgery with respect to the framing given by the pages. We give a detailed answer in* [BRW].

The above formulation allows us to write down an explicit Morse function supporting a compatible Weinstein structure on a PANLF. In [BRW] it will be shown that any compatible Weinstein structure on the 4-manifold, i.e., for which the fibers are almost complex, is homotopic to this one.

4. EXAMPLES OF FILLING CLASSIFICATIONS

In this section, we review several known examples and reinterpret them using the language of spinal open books and nearly Lefschetz fibrations to classify their symplectic fillings. Our primary classification tool is Theorem B. In what follows, we outline the strategy for classifying fillings of (M, ξ) supported by a uniform planar spinal open book (π_Σ, π_P), where the page, monodromy, and vertebra are denoted by $(P, \phi, \Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_r)$. Note that since we deal with uniform spinal open books, all pages should be diffeomorphic, but the monodromies could be different. Let m be the number of connected components of the paper M_P and $\phi = (\phi_1, \ldots, \phi_m)$. Here, as explained in §2.3, we assume that we already identify all pages with a single "model surface" P. The strategy goes as follows:

(i) First, list all possible bases $B \in \mathcal{B}$ such that (π_Σ, π_P) is uniform with respect to B. To find such B, we need to use Definition 2.1 that the number of boundary components of B is the same as the number of connected components of paper M_P, namely m, and each vertebra is (possibly branched) covering of B. These conditions restrict the possible topology of B due to the Riemann–Hurwitz formula:

$$\chi(\Sigma_i) = n \cdot \chi(B) + \sum_{p \in \Sigma}(e_p - 1),$$

where n is the degree of the covering map, e_p is the ramification index at p.

(ii) For each $B \in \mathcal{B}$, find all positive admissible factorizations of ϕ with respect to a monodromy representation $\rho\colon \pi_1(B) \to \mathrm{Mod}(P^c)$ as follows (where P^c is obtained from P by capping off all boundary components corresponding to non-trivial spines):

Suppose each branched covering map $\pi_i : \Sigma_i \to B$ has n_i branch points. Let $n_B = n_1 + \cdots n_r$ be the total number of branch points. This will be the number of exotic fibers in the PANLF induced from the spinal open book and a monodromy factorization.

Since the spinal open book is uniform, for each boundary component ∂_j of B, there exists a corresponding paper component with monodromy ϕ_j that intersects the boundary component $\pi_i^{-1}(\partial_j)$ of Σ_i for every $i = 1, \dots, r$.

Now for each $j = 1, \dots, m$, factorize $\phi_j \circ \rho(\partial_j)^{-1}$ into a product containing boundary interchanges, and positive Dehn twists about homologically essential curves on the page. In fact, each branched cover $\pi_i\colon \Sigma_i \to B$ determines the boundary interchange maps in a factorization. The total number of boundary interchanges across $\phi_1 \circ \rho(\partial_1)^{-1}, \dots, \phi_m \circ \rho(\partial_m)^{-1}$ should be n_B. Repeat this process for all possible monodromy representations ρ.

(iii) As a consequence, we obtain a list of symplectic fillings of (M, ξ), each described as a PANLF over a base B with n_B exotic fibers, as B ranges over $\mathcal{B}$. Distinct PANLFs may potentially support equivalent symplectic structures, requiring us to account for double-counting. See Theorem 4.4 for an example where different monodromy representations induce equivalent symplectic fillings. However, we still lack an example in which two different bases induce equivalent symplectic fillings.

Note that the symplectic filling corresponding to a factorization above is a PANLF over B with fiber P having n_B exotic fibers and sigular fibers along the curves associated to Dehn twists. Each connected component of paper M_P, a P-bundle over S^1 with monodromy ϕ_j, corresponds to a subbundle of this PANLF restricted to the boundary component ∂_j of B.

In general, the classification problem using the machinery of spinal open books consists of two main parts. First, we seek a planar spinal open book decomposition supporting the contact structure under consideration. Second, we study the monodromy factorization problem in the framed mapping class group. The following examples provide a glimpse of how these procedures are carried out in practice. In many cases, the procedure is much simpler, as the base is a disk (so ρ is trivial), or P has only one component.

4.1. **The Stein fillable contact structure on the 3-torus.** The first example is the Stein fillable contact structure on $\mathbb{T}^3$, which is arguably the original motivation for spinal open book decompositions. Then, we consider the examples of other torus bundles and finally give some outlook on contact 3-manifolds supported by more general higher genus open book decompositions.

Recall from §2.4.1 that there exists a unique Stein fillable contact structure ξ_0 on $\mathbb{T}^3$. The classification of symplectic fillings for $(\mathbb{T}^3, \xi_0)$ is due to Wendl [Wen10].

Theorem 4.1 ([Wen10]). *There exists a unique minimal symplectic filling of* $(\mathbb{T}^3, \xi_0)$ *which is given by* $\mathbb{D}^2 \times \mathbb{T}^2$ *equipped with the product symplectic form.*

Here, we provide an alternate proof of Theorem 4.1 using spinal open book decompositions, following the strategy outlined above.

(i) In §2.4.1, we construct a spinal open book $\pi = (\pi_\Sigma, \pi_P)$ supporting $(\mathbb{T}^3, \xi_0)$ that consists of an annulus page P and two paper components with monodromies $\phi_1 = \phi_2 = \mathrm{id}$, and two annular vertebrae Σ_1, Σ_2. We first need to list all possible bases $B \in \mathcal{B}$ such that the spinal open book is uniform with respect to B. Recall from Definition 2.1 that the number of connected components of ∂B is the same as the number of connected components of the paper M_P. Since our M_P has two connected components, we know that B must have two boundary components. According to Definition 2.1 again, there exists a (possibly branched) covering map $\pi_i \colon \Sigma_i \to B$ for $i = 1, 2$. Since both Σ_1 and Σ_2 are annuli, B must also be an annulus. Note that the covering maps do not have a branch point since both B and Σ are annuli. Therefore, we have $\mathcal{B} = \{A\}$ where $A = S^1 \times I$. In general, $\mathcal{B}$ can contain more than one element, and we will see such an example in Theorem 4.10.

(ii) Now, we need to find all possible positive admissible factorizations of ϕ_1 and ϕ_2, with respect to (B, ρ), where $\rho \colon \pi_1(B) \to \mathrm{Mod}(P)$ is a monodromy representation. Since P is an annulus, we have $\mathrm{Mod}_\partial(P) = \langle T_c \rangle$, where c is the core of P. Let ∂_1 and ∂_2 be the boundary components of B corresponding to the paper components with ϕ_1 and ϕ_2, respectively. Since ∂_2 is homotopic to $-\partial_1$, we have

$$\rho(\partial_2) = \rho(-\partial_1) = \rho(\partial_1)^{-1}$$

Now, let $\rho(\partial_1) = T_c^k$ for some $k \in \mathbb{Z}$. Then $\rho(\partial_2) = T_c^{-k}$.

Exercise 4.2. Show that $\rho(\partial_1)$ cannot contain a boundary interchange map.
(Hint: Proposition 4.5 may be helpful.)

Now we factorize monodromies $\phi_1 \circ \rho(\partial_1)^{-1} = T_c^{-k}$ and $\phi_2 \circ \rho(\partial_2)^{-1} = T_c^k$ into a product of positive Dehn twists along homologically essential curves on P. There is no boundary interchange since π_1 and π_2 do not have a branch point. However, this is impossible unless $k = 0$ since otherwise, either k or $-k$ would be negative.

Therefore, $k = 0$ and the monordomy factorizations of $\phi_1 \circ \rho(\partial_1)^{-1}$ and $\phi_2 \circ \rho(\partial_2)^{-1}$ are both id.

(iii) $\phi = (\phi_1, \phi_2)$ admits a unique positive admissible monoromy factorization, and hence $(\mathbb{T}^3, \xi_0)$ admits a unique minimal symplectic filling that admits a bordered Lefschetz fibration over an annular base with an annular fiber without singular and exotic fibers. Therefore, it is diffeomorphic to $A \times A$ and we have

$$\begin{aligned} A \times A &= (S^1 \times I) \times (S^1 \times I) \\ &\cong (I \times I) \times (S^1 \times S^1) \\ &\cong \mathbb{D}^2 \times \mathbb{T}^2 \end{aligned}$$

Exercise 4.3. Show that the product symplectic structure of $\mathbb{D}^2 \times \mathbb{T}^2$ is deformation equivalent to the product symplectic structure of $A \times A$.

4.2. **Symplectic fillings of torus bundles.** The next examples are parabolic torus bundles over S^1. Recall from §2.4.3 that a torus bundle T_A is **parabolic** if $|\mathrm{tr}(A)| = 2$, and it is **positive** if A is conjugate to $A_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ for some $k \in \mathbb{Z}$ and **negative** if A is conjugate to $-A_k$; there are denoted by $T_\pm(k)$. Our main interest here is in rotational contact structures on these bundles, which were studied in [DG01, VHM07, DL18, LHMW20, MRW25]. We again follow the strategy outlined at the beginning of this section to classify the symplectic fillings of these contact structures.

Theorem 4.4 (Wendl–Lisi–Van Horn-Morris [LHMW20], Min–Roy–Wang [MRW25])**.** *Suppose $k \in \mathbb{Z}$. Then the following statements hold.*

1. *$(T_+(k), \xi_2)$ is strongly fillable if and only if $k \geq 0$, and for $k \geq 0$, it admits a unique Stein filling up to symplectic deformation equivalence.*
2. *$(T_-(n), \xi_1)$ is strongly fillable if and only if $k \geq -4$, and for $k \geq -4$, it admits a unique Stein filling up to symplectic deformation equivalence.*
3. *all other rotational contact structures, $(T_+(n), \xi_{2k})$ for $k > 1$ and $(T_-(n), \xi_{2j+1})$ for $j \geq 1$, do not admit a strong filling.*

The first part of Theorem 4.4 demonstrates that two different monodromy representations, ρ_1 and ρ_2, can produce equivalent symplectic structures, and the second part provides an example that a factorization of monodromy contains boundary interchange maps. To prove the theorem, we need the following proposition, which provides a factorization of two boundary interchange maps as a product of Dehn twists.

Proposition 4.5. *Consider a pair of pants with two arcs α and β joining inner boundary components as shown in the left drawing of Figure 11. Then make additional boundary component in the middle of the disk as shown in the left drawing of Figure 10. Then*

- *For a disk with 3 punctures, $\tau_\beta \circ \tau_\alpha = T_{\partial_1} T_{\partial_3} T_{\partial_4}^{-2} T_c^{-2}$, where c is a curve enclosing ∂_2 and ∂_3. See the right drawing of Figure 10.*
- *For a pair of pants, $\tau_\beta \circ \tau_\alpha = T_{\partial_1} T_{\partial_2}^{-2} T_{\partial_3}^{-2}$. See the right drawing of Figure 11.*

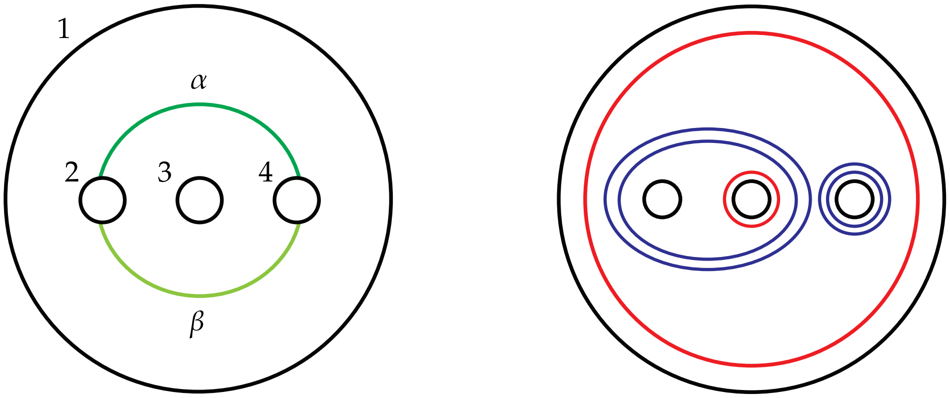

FIGURE 10. Left: Two consecutive boundary interchanges $\tau_\beta\tau_\alpha$ of the inner boundary components along two non-intersecting arcs α and β. Right: A monodromy factorization of $\tau_\beta\tau_\alpha$.

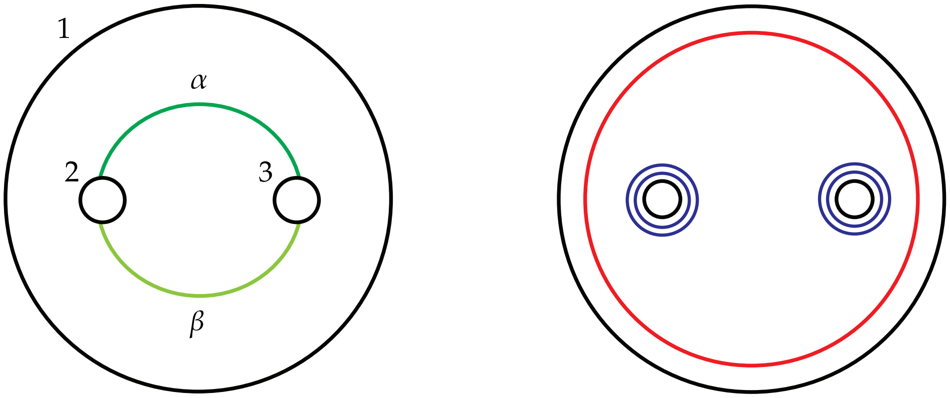

FIGURE 11. Left: Two consecutive boundary interchanges $\tau_\beta\tau_\alpha$ of the inner boundary components along two non-intersecting arcs α and β. Right: A monodromy factorization of $\tau_\beta\tau_\alpha$.

Exercise 4.6. Prove Proposition 4.5.

Proof of Theorem 4.4. The third part of the theorem is immediate from the fact that the contact structures under consideration contain Giroux torsion. We now consider the first part.

(i) We begin by finding a planar spinal open book decomposition supporting $(T_+(k), \xi_2)$. According to Van Horn-Morris [VHM07, Proof of Theorem 4.3.1], the word for $(T_+(k), \xi_2)$ is $(aba)^{-4}a^k$. Consequently, we obtain a spinal open book with two annular vertebrae, each corresponding to a relative open book with word $(aba)^{-2}$, one annular page with identity monodromy, and another annular page with monodromy

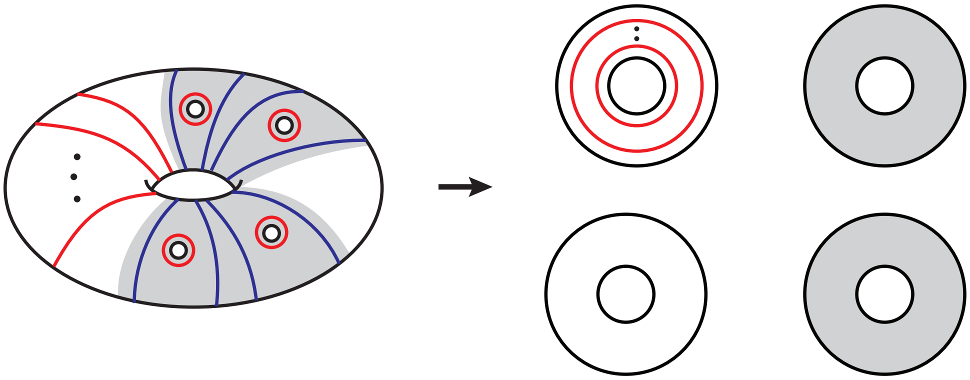

FIGURE 12. Left: An open book decomposition supporting $(T_+(k), \xi_2)$. Each gray region represents a relative open book with the word $(aba)^{-2}$. Right: A spinal open book decomposition supporting $(T_+(k), \xi_2)$. The gray annuli are vertebrae corresponding to the word $(aba)^{-2}$.

T_c^k, where c is the core of the annulus (see Figure 12). Thus, we have pages components, both of which are annuli. Here we again denote them by P as a 'model page' that identifies two page components. See §2.3 for more details. Also we have monodromy $\phi = (\phi_1, \phi_2)$, where $\phi_1 = T_c^k$ and $\phi_2 = \mathrm{id}$, and vertebrae $\Sigma = \Sigma_1 \sqcup \Sigma_2$, both of which are annuli.

We then list all possible bases $B \in \mathcal{B}$ such that the spinal open book is uniform with respect to B. Recall from Definition 2.1 that the number of connected components of ∂B is the same as the number of connected components of the paper M_P. Since our M_P has two connected components, we know that B must have two boundary components. According to Definition 2.1 again, there exists a (possibly branched) covering map $\pi_i \colon \Sigma_i \longrightarrow B$ for $i = 1, 2$. Since both Σ_1 and Σ_2 are annuli, B must also be an annulus. Note that the covering maps do not have a branch point since both B and Σ are annuli. Therefore, we have $\mathcal{B} = \{A\}$ where $A = S^1 \times I$.

(ii) Now, we need to find all possible positive admissible factorizations of ϕ_1 and ϕ_2, with respect to (B, ρ), where $\rho \colon \pi_1(B) \longrightarrow \mathrm{Mod}(P)$ is a monodromy representation. Since P is an annulus, we have $\mathrm{Mod}_\partial(P) = \langle T_c \rangle$. Let ∂_1 and ∂_2 be the boundary components of B corresponding to the paper components with ϕ_1 and ϕ_2, respectively. Since ∂_2 is homotopic to $-\partial_1$, we have

$$\rho(\partial_2) = \rho(-\partial_1) = \rho(\partial_1)^{-1}$$

Now, let $\rho(\partial_1) = T_c^i$ for some $i \in \mathbb{Z}$. Then $\rho(\partial_2) = T_c^{-i}$. We factorize monodromies $\phi_1 \circ \rho(\partial_1)^{-1} = T_c^{k-i}$ and $\phi_2 \circ \rho(\partial_2)^{-1} = T_c^i$ into a product of positive Dehn twists along homologically essential curves on P. There is no boundary interchange since π_1 and π_2 do not have a branch point. Now if $k < 0$, this is impossible since either $k - i$ or i would be negative. Therefore, $k \geq 0$ and $k = 0$ is already dealt in Theorem 4.1.

Finally, if $k > 0$, then both $k - i$ and i are positive as long as $0 \leq i \leq k$. Therefore, there are $k+1$ possible monomromy representations $\rho_0, \ldots, \rho_k$ such that $\rho_i(\partial_1) = T_c^i$.

Then, for each $i = 0, \ldots, k$, we have unique positive admissible factorization

$$\phi_1 \circ \rho_i(\partial_1)^{-1} = T_c^{k-i} \text{ and } \phi_2 \circ \rho_i(\partial_2)^{-1} = T_c^i.$$

(iii) From the factorizations above, we obtain $k+1$ symplectic fillings from PANLF over B without exotic fibers and $k = (k-i) + i$ singular fibers. That is, since B is an annulus, they can be obtained from an honest symplectic fiber bundle over A with fiber A and monodromy representation ρ_i for $i = 0, \ldots, k$, and then attaching k Weinstein 2-handles along the curves associated with the Dehn twists.

Fianlly, we need to check for duplications. To do so, we first note that any two oriented A-bundles over A are isomorphic to each other, related by a change of trivialization. To see this, consider an A-bundle over A with ρ_0, which is a trivial bundle $A \times A$. Then the following diffeomorphism induces a bundle isomorphism between two bundles with monodromy representations ρ_0 and ρ_i:

$$F_i \colon A \times A \to A \times A, \quad F_i(\theta, r, \phi, s) = (\theta, r, \phi - i\theta s, s)$$

That is, after changing trivializations, all bordered Lefschetz fibrations become equivalent and they induce a spinal open book on their boundary with two annular pages, one with monodromy T_c^k and the other with the identity monodromy. Thus $(T_+(k), \xi_2)$ admits a unique minimal symplectic filling if and only if $k \geq 0$.

Next, we consider the second part. Again, we begin by finding a spinal open book decompostion supporting $(T_-(k), \xi_1)$.

(i) According to Van Horn-Morris [VHM07, Theorem 4.3.1], a word for $(T_-(k), \xi_1)$ is $(aba)^{-2}a^k$. Therefore, we can convert it into a spinal open book decomposition with one annular vertebra Σ and one annular page P with monodromy $\phi = T_c^k$ where c is the core of the annular page.

We then list all possible bases $B \in \mathcal{B}$ such that the spinal open book is uniform with respect to B. Since our paper M_P has a single connected component, we know that B must have one boundary component. Then the existence of a (possibly branched) covering map $\pi \colon \Sigma \to B$ for $i = 1, 2$ restricts B to be a disk. Note that the covering maps has two branch points. Therefore, we have $\mathcal{B} = \{\mathbb{D}^2\}$.

(ii) Now, we need to find all possible positive admissible factorizations of ϕ, with respect to (B, ρ), where $\rho \colon \pi_1(B) \to \mathrm{Mod}(P^c)$ is a monodromy representation and P^c is the sphere obtained by capping off the boundary components of P (recall that we need to cap off the boundary components corresponding to non-trivial vertebrae). Since B is a disk, ρ must be trivial so we only need to factorize $\phi = T_c^k$. Since there are two branch points, a factorization should contain two boundary interchange maps. According to Propositon 4.5, two consecutive boundary interchanges can be factorized into Dehn twists about the boundary components, i.e., $\tau_\beta \tau_\alpha = T_{\partial_1} T_{\partial_2}^{-2} T_{\partial_3}^{-2}$ as shown in Figure 11. Once we cap off the outer boundary component ∂_1, we obtain two consecutive boundary interchanges of an annulus, and its factorization becomes $\tau_\beta\tau_\alpha = T_{\partial_c}^{-4}$ as ∂_2 and ∂_3 become homotopic to the core c of the annnulus. In our setting, however, two arcs α and β may not be homotopic rel boundary, but we still have a Hurwitz equivalent factorization.

Exercise 4.7. Let α and β be two arcs joining two boundary components of an annulus that are not homotopic rel boundary. Show that τ_α and τ_β are related by a framing conjugation (see Deifiniton 2.13).

Hence we have the following factorization of ϕ:

$$\phi = T_c^k = T_c^{k+4} \circ \tau_\beta \circ \tau_\alpha$$

(iii) Therefore, ϕ admits a unique positive admissible factorization if and only if $k \geq -4$, and this proves the second part of the Theorem.

□

Combined with the results regarding additional tight contact structures on parabolic torus bundles in [LHMW20, Chr21], Theorem 4.4 completes the classification of symplectic fillings of every contact structure on all parabolic torus bundles.

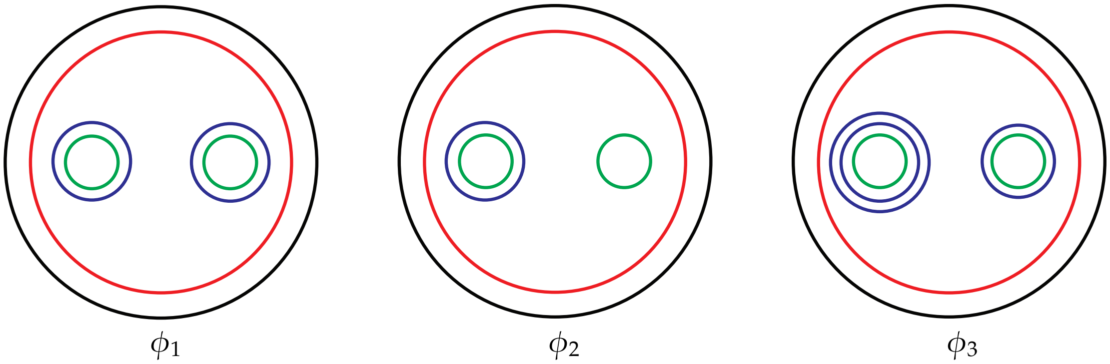

FIGURE 13. The monodromies of the spinal open books supporting the π-twisting rotational elliptic torus bundles. The green boundary components meet the annulus vertebra.

Next, we consider the elliptic torus bundles with π-twisting rotational contact structures.

Theorem 4.8 (Min-Roy-Wang [MRW25]). *Any rotational contact structure on an elliptic torus bundle with π-twisting admits a unique Stein filling up to symplectic deformation equivalence.*

Exercise 4.9. Prove Theorem 4.8.
(Hint: Use Exercise 2.17. See also Figure 13 and Figure 14 for corresponding spinal open books.)

4.3. **Further examples.** Here, we present Stein fillings of some contact 3-manifold supported by a planar spinal open book decomposition that can be uniform with respect to different bases.

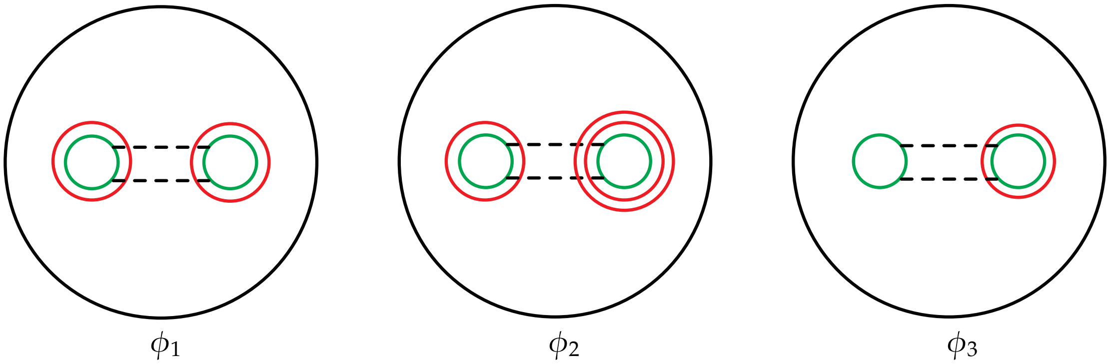

FIGURE 14. Positive admissible monodromy factorizations for π-twisting elliptic torus bundles, with boundary interchanges along the dotted arcs.

This contact 3-manifold is obtained by taking the boundary of the quotient of a product symplectic manifold. We first recall that $\Sigma_{g,n}$ is a genus g surface with n boundary components. Let ψ_1 and ψ_2 be involutions of $\Sigma_{2,2}$ arising as the deck transformations of the double branched covers $\phi_1 : \Sigma_{2,2} \to \Sigma_{1,2}$ and $\phi_2 : \Sigma_{2,2} \to \Sigma_{0,2}$, where ϕ_1 is unbranched and ϕ_2 has four simple branch points. Also let σ be an involution $\sigma(s,t) = (-s,-t)$ of $\Sigma_{0,2} = [-1,1] \times S^1$. Then for $i = 1,2$, the following Weinstein domains

$$E_i = (\Sigma_{2,2} \times \Sigma_{0,2})/(z,w) \sim (\psi_i(z), \sigma(w)) \tag{5}$$

induce the same boundary contact 3-manifold (M,ξ), and it is supported by a spinal open book with two annular pages such that each page has a boundary interchange map as the monodromy, and a single vertebra $\Sigma_{2,2}$. Thus we can denote the page monodromies by $\phi = (\phi_1, \phi_2) = (\tau_\alpha, \tau_\beta)$, where α is an arc joining two boundary components of the first page, and β is an arc joining two boundary components of the second page.

Although it is challenging to classify all symplectic fillings of (M,ξ) since lisitng all possible monodromy representations is difficult, we claim that (M,ξ) admits two types of fillings: the first is a bordered Lefschetz fibration (including E_1), and the second is E_2.

Theorem 4.10 (Wendl-Lisi-Van Horn-Morris [LHMW20], Min-Roy-Wang [MRW25])**.** *Consider the contact 3-manifold (M,ξ) above. The Stein fillings of this manifold are either bordered Lefschetz fibrations over $\Sigma_{1,2}$ with $\Sigma_{0,2}$-fiber without singular fibers, or a PANLF over $\Sigma_{0,2}$ with $\Sigma_{0,2}$-fiber, and four singular fibers.*

Proof. We again follow the strategy outlined at the beginning of the section. Since pages have one boundary component with muliplicity 2, there exists a (possibly branched) 2-fold cover $\pi_1 \colon \Sigma_{2,2} \to B$ that induces a 2-fold honest cover on the boundaries. According to the Riemann–Hurwitz formula, B should be either $\Sigma_{1,2}$ or $\Sigma_{0,2}$.

In the case where $B = \Sigma_{1,2}$, there are no branch points due to the Riemann–Hurwitz formula. Thus there are no exotic fibers, and hence all symplectic fillings are bordered Lefschetz fibrations over $\Sigma_{1,2}$. These can be constructed from a $\Sigma_{0,2}$-bundle over $\Sigma_{1,2}$ with monodromy representation $\rho\colon \Sigma_{1,2} \to \Sigma_{0,2}$, followed by attaching Weinstein 2-handles along the curves associated with singular fibers that arise from a positive admissible factorization of the page monodromies. We obtain a family of such fillings by varying the monodromy representation.

In the case where $B = \Sigma_{0,2}$, there are four branch points due to the Riemann–Hurwitz formula. Thus all symplectic fillings contain 4 exotic fibers in their completions, and hence are PANLF over an annular base. These can be constructed from a $\Sigma_{0,0}$-bundle over $\Sigma_{0,2}$ with monodromy representation $\rho\colon \Sigma_{0,2} \to \Sigma_{0,0}$ (which is always trivial), followed by removing a positive multisection, that creates two boundary components on the pages (hence they become $\Sigma_{0,2}$), and attaching Weinstein 2-handles along curves associated with singular fibers that come from a positive admissible factorization of the page monodromies. Since there are four branch points, there should be four boundary interchange maps in the page monodromies. Thus the monodromy of one page component of the induced spinal open book should involve a boundary interchange, while the monodromy of the other component should involve three boundary interchanges. Since we have $\phi_1 = \tau_\alpha$ and $\phi_2 = \tau_\beta$, and according to Proposition 4.5, we factorize ϕ_2 as follows:

$$\phi_2 = \tau_\beta = \tau_\beta \circ (\tau_\beta^2 \circ T_c^4) = \tau_\beta^3 \circ T_c^4,$$

where c is the core of the annular page.

Exercise 4.11. Prove that the fatorization above is the unique positive admissible factorization of ϕ up to a permutation of the pages.

Since there exists a unique trivial monodromy representation and a unique positive admissible factorization of the page monodromy, there exists a unique minimal symplectic filling. This filling is PANLF with four exotic fibers and four singular fibers that have c as their vanishing cycles.

Exercise 4.12. Prove that E_2 is deformation equivalent to the symplectic filling we construct.

□

5. FUTURE QUESTIONS

In this section, we outline possible future directions of research, and some examples that will be interesting to work out in detail.

5.1. **Classifying fillings of higher genus open books.** First, we provide a motivating family of examples of open book with higher genus pages. These examples illustrate both the usefulness and complexity of using spinal open books to classify symplectic fillings of contact 3-manifolds supported by higher genus open books.

Specifically, the following family of open books shows how certain open books with arbitrarily high genus can be converted into planar uniform spinal open books, which makes understanding their fillings a more tractable problem. However these examples also highlight the computational difficulty of the classification problem in this setting.

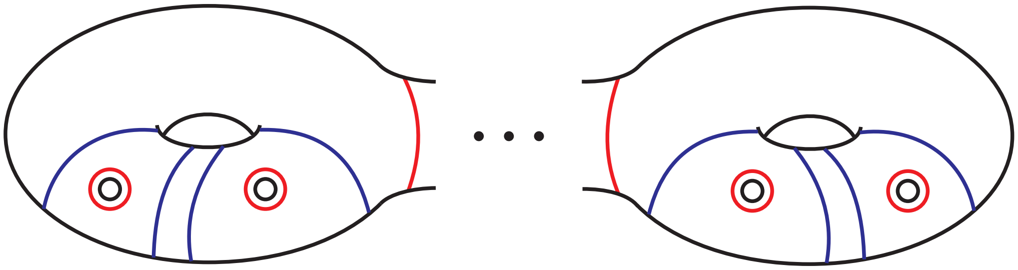

FIGURE 15. A high genus open book decomposition which can be turned into a planar spinal open book.

Example 5.1. Consider the family of open books shown in Figure 15. For every genus $g \geq 2$, this represents a Stein fillable contact 3-manifold that admits at least one Stein filling which is a Lefschetz fibration over the disk with genus g fibers, and is at the same time a nearly Lefschetz fibration over the disk with planar fibers. The Stein filling is originally given by [DL18]. In fact, the proof of Proposition 3.2 in [DL18] shows that this open book monodromy admits a positive factorization, which corresponds to a Stein filling that admits a Lefschetz fibration with genus g fibers. On the other hand, we can replace the 'handle' parts – relative open books corresponding to $(aba)^{-2}$ – with annular vertebrae by Proposition 2.16, and obtain a planar uniform spinal open book supporting the same contact manifold. Theorem B then shows that all Stein fillings must have the structure of a PANLF over a disk with planar fibers.

Exercise 5.2. Use Theorem B to show that the contact 3-manifold in Example 5.1, for $g = 2$, has at least two minimal symplectic fillings: one with six singular fibers, and one with seven singular fibers.

As the solution to the above exercise suggests (the reader can refer [MRW25] for the answer), the filling classification problem deals with complicated configurations of arcs along which boundary interchanges can happen. A resolution to the following question will thus provide handy computational tools in using the technology of planar spinal open books.

Question 5.3. *What is the full classification of minimal symplectic fillings of the contact* 3-*manifolds in Example 5.1?*

5.2. **Giroux correspondence for spinal open books.** It is shown in [LHMW18, Theorem 1.7] that the space of Giroux forms supporting a spinal open book is contractible. However, since vertebrae can have complicated topology, it is likely that the notion of **stabilization** of open books needs to be generalized, for example, including stabilizations on vertebrae.

Question 5.4. *What is the correct definition of stabilization of a spinal open book for which the Giroux correspondence holds?*

In addition, we need to generalize the notion of right- and left-veering [HKM07] to the spinal setting.

Question 5.5. *Is there a notion of right- and left-veering, which characterizes overtwistedness for spinal open book decompositions?*

We can also ask whether additional symplectic invariants arise that help distinguish fillings beyond topological obstructions, such as the number of vanishing cycles.

Question 5.6. *Are there diffeomorphic but not symplectic deformation equivalent fillings for a fixed contact* 3*-manifold?*

5.3. **Support genus problem.**

The first obstruction to the planarity of an open book is given by Theorem 1.8, using McDuff's characterization of ruled surfaces [McD90].

By Theorems 1.5, A and 1.10, if a contact 3-manifold admits a strong but not Stein filling, then it is not supported by a planar (uniform spinal) open book. This is because for planar (uniform spinal) open books, all of its minimal strong fillings are deformation equivalent to positive allowable (nearly) Lefschetz fibrations, which are supported by Stein structures.

Generalizing this, Wendl gives an obstruction to planar uniform spinal open books in ECH.

Theorem 5.7. [Wen13, Theorem 6] *If (M, ξ) is supported by a planar uniform spinal open book, then the ECH contact invariant is at the bottom of a U-tower.*

In light of this, we naturally ask whether a similar obstruction exists in the Heegaard Floer setting.

Question 5.8. *Is there a Heegaard Floer obstruction to a contact three-manifold being supported by a planar spinal open book?*

One way to compute the Heegaard Floer contact invariant is by converting a supporting open book (Σ, ϕ) into a Heegaard diagram. This leads to the following question.

Question 5.9. *Is there a natural way to convert to spinal open books to Heegaard diagrams?*

Recall that there is a description for the relative open book for a spine component with annulus vertebra (see Proposition 2.16). One approach to answering Question 5.9 is to find relative open books for spine components with vertebrae of different topology.

Note that for spinal open books arising as the boundary of PANLFs, there is an algorithm to convert the spinal open book to a regular open book. This is in theory as a result of Theorem 1.10. In more words, since Exercise 3.9 shows how to produce a Kirby-Weinstein diagram for the manifold, we can convert that into an open book following the algorithm in [Avd13]. However this may not be the most efficient. It is certainly worth asking if there is a direct way to produce regular open books of "least complexity" out of spinal open books.

Question 5.10. *What is an efficient way to produce regular open books out of spinal open books?*

One can hope these tools will prove beneficial in answering the following long-standing open question about the support genus of open books. While we know that there are obstructions to contact manifolds being supported by planar open books, we know very little beyond.

Question 5.11 (Support genus problem)**.** *Is there a contact* 3*-manifold with support genus two, i.e., there is no genus one open book supporting it?*

5.4. **Connections to 4-manifold topology and near symplectic structures.** Lefschetz fibrations and pencils provide a powerful characterization of symplectic 4-manifolds. Relaxing the geometric condition, Etnyre and Fuller [EF06] showed that an "achiral Lefschetz fibration" on a 4-manifold can be endowed with a near-symplectic structure, i.e., a symplectic structure on the complement of a collection of embedded S^1's. They also showed that any smooth, closed, oriented 4-manifold admits an achiral Lefschetz fibration after surgery on a framed circle. Simply put, achiral Lefschetz fibrations allow topological vanishing cycles where the associated monodromy is a left-handed Dehn twist.

One can wonder whether similar notions exist for nearly Lefschetz fibrations, and if they can provide tools for understanding near-symplectic structures. A natural thought is to introduce the clockwise or negative boundary interchange mapping class, to allow for "topological exotic fibers".

Question 5.12. *What is the correct notion of an achiral nearly Lefschetz fibration? Does every smooth, closed, oriented 4-manifold admit one? Are they compatible with near symplectic structures?*

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