Exotic 4-manifolds with boundary

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4-manifolds with boundary

• Goal: studying smooth structures on 4-manifolds with boundary.

• Try to figure out which 3-manifolds bound a compact 4-manifold with infinitely many smooth structures.
Strategy

- Find/build a 4-manifold which $Y$ bounds (embeds).
- Modify the 4-manifold keeping the homeomorphism type.
- Use smooth invariants ($\text{SW}$, $\text{HF}$, genus, ...) to distinguish them.
Heegaard Floer invariants

- $W$ is a 4-manifold with $b_2^+ > 1$
- An admissible cut is a separating hypersurface $N$ satisfying
  \[ b_2^+(W_1), b_2^+(W_2) \geq 1 \]
  \[ \delta : H^1(N) \to H^2(W, \partial W) \text{ vanishes} \]

\[
F_{W,s}^{mix} := F_{W_2,s|_{W_2}}^+ \circ \tau \circ F_{W_1,s|_{W_1}}^-
\]
Heegaard Floer invariants

- $X$ is a 4-manifold with connected boundary
- $b = (b_1, \ldots, b_n)$ is a basis of $H^2(X, \partial X)$
- (Juhasz-Zemke) Ozsvath-Szabo polynomial is defined as

$$
\Phi_{X;b} := \sum_{s \in \text{Spin}^c(X)} F_{X,s}^{\text{mix}}(\theta^-) \cdot z_1^{i_*(s-s_0) \cup b_1,[X,\partial X]} \cdots z_n^{i_*(s-s_0) \cup b_n,[X,\partial X]} $$
Heegaard Floer invariants

- $\Phi_{X;b}$ is an invariant of $X$ up to automorphisms of $HF^+(\partial X)$

- If $X$ is a closed 4-manifold, use $X \setminus B^4$
Concordance surgery

• A generalization of Fintushel-Stern knot surgery.
Concordance surgery

• Let $X$ be a 4-manifold containing an embedded torus $T$ with a trivial normal bundle.

• Consider a homology sphere $Y$ and a self concordance $C = (Y \times I, A, K)$. Glue the ends of $(Y \times I, A)$ to obtain $(Y \times S^1, T')$. 


Concordance surgery

• We obtain

\[ X_C := X \setminus N(T) \cup_\phi Y \times S^1 \setminus N(T') \]

where \( \phi : \partial (X \setminus N_T) \to \partial W_C \) is a diffeomorphism sending \( \partial D^2 \)
• in \( N(T) \) to the Seifert longitude of \( K \).
Graded Lefschetz number

(Juhász-Maregon) For a self-concordance $\mathcal{C} = (Y \times I, A, K)$

$$\hat{F}_{\mathcal{C}, i} : \widehat{HFK}(Y, K, i) \to \widehat{HFK}(Y, K, i)$$

$$\operatorname{Lef}_\mathcal{C}(z) := \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{Tr}(\hat{F}_{\mathcal{C}, i}) \cdot z^i$$
Concordance surgery formula

• (Juhasz-Zemke) If $X$ is a closed 4-manifold containing a homologically non-trivial torus with trivial normal bundle,

$$\Phi_{X_c; b} = \text{Lef}_C(z) \cdot \Phi_{X; b}$$

• (Etnyre-M-Mukherjee) If $X$ is a 4-manifold with boundary

$$\Phi_{X_c; b} = \text{Lef}_C(z) \cdot \Phi_{X; b}$$

• If $\text{Lef}_C(z)$ and $\text{Lef}_{C'}(z)$ are different and $\Phi_{X; b} \neq 0$, then $X_C$ is not diffeomorphic to $X_{C'}$. 
Main Theorem

A closed oriented 3-manifold $Y$ admits infinitely many simply-connected exotic fillings if

1) $Y$ admits a contact structure with non-vanishing contact invariant in $HF^+(Y)$, or

2) $Y$ is a rational homology 3-sphere embedding into a closed definite 4-manifold as a separating hypersurface.
Proof of (2)

• Suppose $Y$ embeds into a closed negative definite 4-manifold $W$.

• For any $\text{Spin}^c$ structure on $W$, the map

$$F^+_{W,s} : HF^+(S^3, t) \to HF^+(S^3, t)$$

is surjective.
Proof of (2)

• Cut $W$ along $Y$ into $W_1 \cup W_2$. Then the map

$$F_{W_1,s|W_1}^+: HF^+(S^3, t) \to HF^+(Y, s|_Y)$$

satisfies

$$F_{W_1,s|W_1}^+(\theta^+) \neq 0$$
Proof of (2)

- Consider a 4-manifold \( X = K3 \# W_1 \)

- Since \( F_{K3,8}^{\text{mix}}(\theta^-) = \theta^\dagger \), we have
  \[
  F_{K3\#W_1}^{\text{mix}}(\theta^-) \neq 0
  \]

- Now apply the surgery formula on the torus in K3 surface.
Questions

• Given a 3-manifold $Y$ and an element $\eta \in HF^+(Y)$, is there a 4-manifold $X$ such that $F^{mix}_{X,s}(\theta^-) = \eta$?

• Given a 3-manifold $Y$, is there any 4-manifold $X$ such that $F^{mix}_{X,s}(\theta^-) \neq 0$?
Thank you!