Exotic 4-manifolds with boundary

Hyunki Min
with John Etnyre and Anubhav Mukherjee

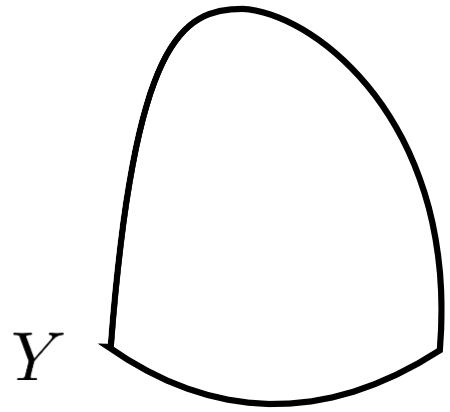
Joint Mathematics Meeting
January 2021

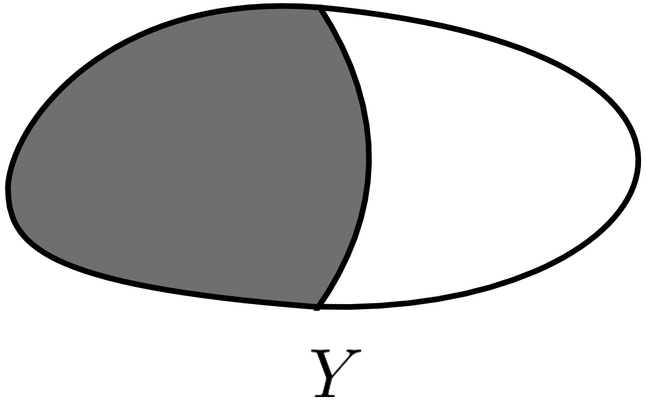
4-manifolds with boundary

- Goal: studying smooth structures on 4-manifolds with boundary.

- Try to figure out which 3-manifolds bound a compact 4-manifold with infinitely many smooth structures.

Strategy

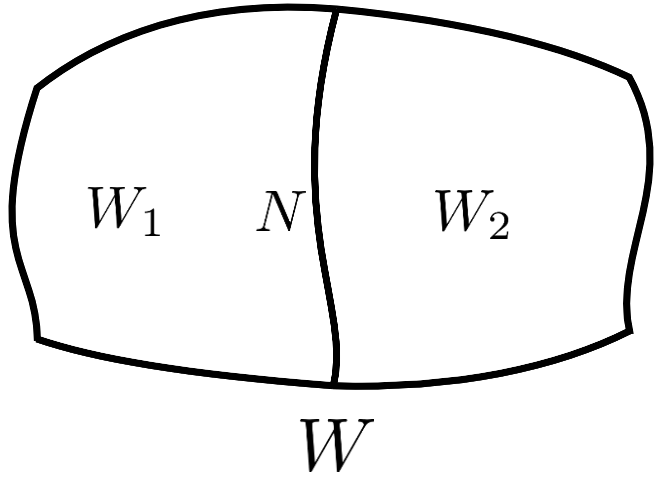




- Find/build a 4-manifold which Y bounds (embeds).
- Modify the 4-manifold keeping the homeomorphism type.
- Use smooth invariants (SW, HF, genus, …) to distinguish them.

Heegaard Floer invariants

- W is a 4-manifold with $b_2^+ > 1$
- An admissible cut is a separating hypersurface N satisfying



$$b_2^+(W_1), b_2^+(W_2) \geq 1$$

$$\delta : H^1(N) \to H^2(W, \partial W)$$ vanishes

$$F^{mix}_{W,s} := F^+_{W_2, s|_{W_2}} \circ \tau \circ F^-_{W_1, s|_{W_1}}$$

Heegaard Floer invariants

- X is a 4-manifold with connected boundary
- $b = (b_1, ..., b_n)$ is a basis of $H^2(X, \partial X)$
- (Juhasz-Zemke) Ozsvath-Szabo polynomial is defined as

$$\Phi_{X;b} := \sum_{s \in \mathrm{Spin}^c(X)} F^{mix}_{X,s}(\theta^-) \cdot z_1^{\langle i_*(s-s_0) \cup b_1, [X, \partial X] \rangle} \cdots z_n^{\langle i_*(s-s_0) \cup b_n, [X, \partial X] \rangle}$$

Heegaard Floer invariants

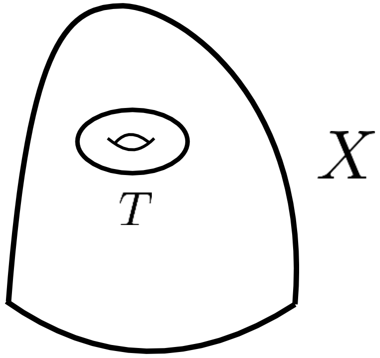
- $\Phi_{X;b}$ is an invariant of X up to automorphisms of $HF^+(\partial X)$
- If X is a closed 4-manifold, use $X \setminus B^4$

Concordance surgery

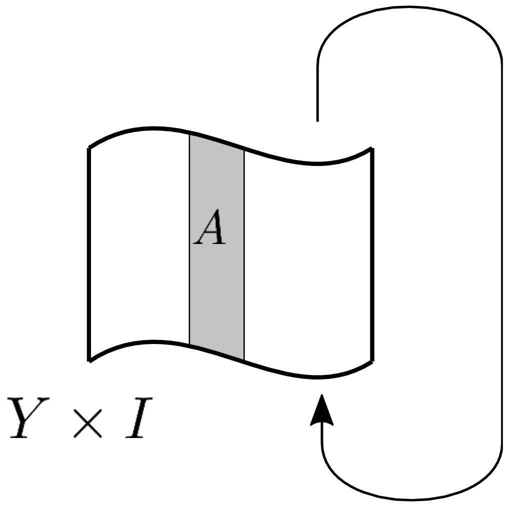
- A generalization of Fintushel-Stern knot surgery.

Concordance surgery

- Let X be a 4-manifold containing an embedded torus T with a trivial normal bundle.



- Consider a homology sphere Y and a self concordance $C = (Y \times I, A, K)$. Glue the ends of $(Y \times I, A)$ to obtain $(Y \times S^1, T')$.

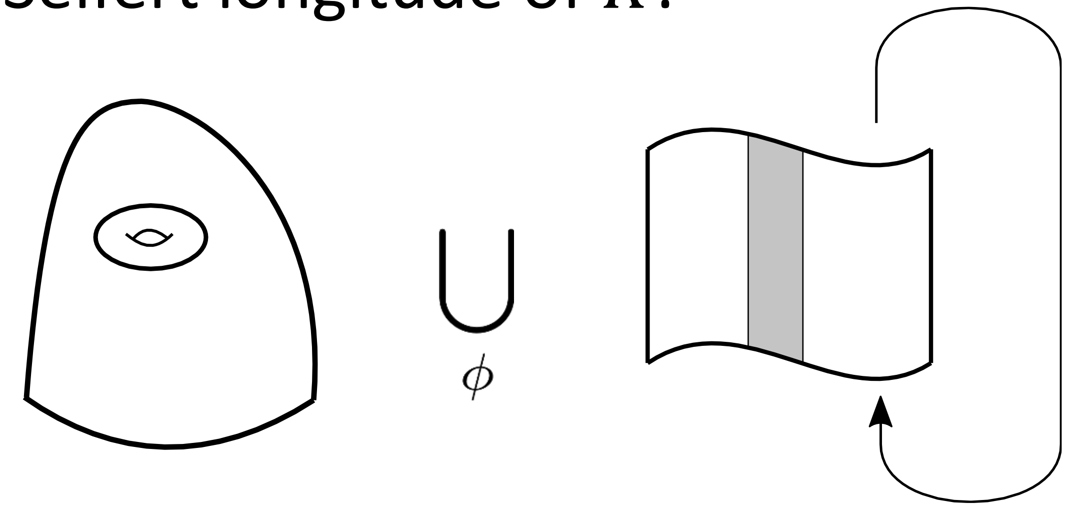


Concordance surgery

- We obtain

$$X_{\mathcal{C}} := X \setminus N(T) \cup_{\phi} Y \times S^1 \setminus N(T')$$

where $\phi : \partial(X \setminus N_T) \to \partial W_{\mathcal{C}}$ is a diffeomorphism sending ∂D^2
- in $N(T)$ to the Seifert longitude of K.



Graded Lefschetz number

(Juhász-Maregon) For a self-concordance $\mathcal{C} = (Y \times I, A, K)$

$$\widehat{F}_{\mathcal{C},i} : \widehat{HFK}(Y, K, i) \to \widehat{HFK}(Y, K, i)$$

$$\mathrm{Lef}_{\mathcal{C}}(z) := \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{Tr}(\widehat{F}_{\mathcal{C},i}) \cdot z^i$$

Concordance surgery formula

- (Juhasz-Zemke) If X is a closed 4-manifold containing a homologically non-trivial torus with trivial normal bundle,

$$\Phi_{X_{\mathcal{C}};b} = \mathrm{Lef}_{\mathcal{C}}(z) \cdot \Phi_{X;b}$$

- (Etnyre-M-Mukherjee) If X is a 4-manifold with boundary

$$\Phi_{X_{\mathcal{C}};b} = \mathrm{Lef}_{\mathcal{C}}(z) \cdot \Phi_{X;b}$$

- If $\mathrm{Lef}_{\mathcal{C}}(z)$ and $\mathrm{Lef}_{\mathcal{C}'}(z)$ are different and $\Phi_{X;b} \neq 0$, then $X_{\mathcal{C}}$ is not diffeomorphic to $X_{\mathcal{C}'}$

Main Theorem

A closed oriented 3-manifold Y admits infinitely many simply-connected exotic fillings if

1) Y admits a contact structure with non-vanishing contact invariant in $HF^+(Y)$, or
2) Y is a rational homology 3-sphere embedding into a closed definite 4-manifold as a separating hypersurface.

Proof of (2)

- Suppose Y embeds into a closed negative definite 4-manifold W.

- For any Spin^c structure on W, the map

$$F^+_{W,s} : HF^+(S^3, t) \to HF^+(S^3, t)$$

  is surjective.

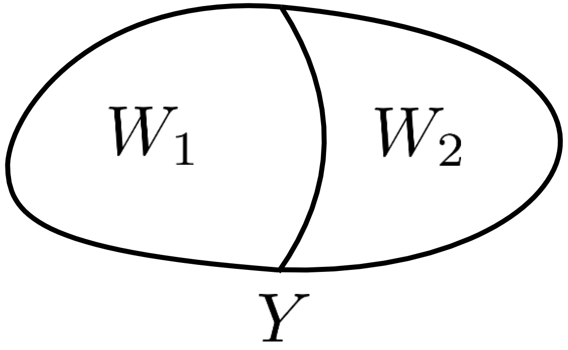
Proof of (2)

- Cut W along Y into $W_1 \cup W_2$. Then the map

$$F^+_{W_1, s|_{W_1}} : HF^+(S^3, t) \to HF^+(Y, s|_Y)$$

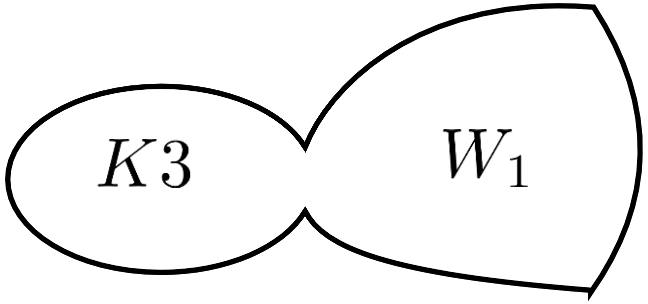
satisfies

$$F^+_{W_1, s|_{W_1}}(\theta^+) \neq 0$$



Proof of (2)

- Consider a 4-manifold $X = K3\#W_1$



- Since $F^{mix}_{K3,s}(\theta^-) = \theta^+$, we have

$$F^{mix}_{K3\#W_1}(\theta^-) \neq 0$$

- Now apply the surgery formula on the torus in K3 surface.

Questions

- Given a 3-manifold Y and an element $\eta \in HF^+(Y)$, is there a 4-manifold X such that $F^{mix}_{X,s}(\theta^-) = \eta$?

- Given a 3-manifold Y, is there any 4-manifold X such that $F^{mix}_{X,s}(\theta^-) \neq 0$?

Thank you!