Cabling Legendrian knots

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Legendrian knots in \((\mathbb{R}^3, \xi_{std})\)

- A standard contact structure on \(\mathbb{R}^3\) is a plane field
  \[\xi = \ker dz - ydx\]
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Classical invariants

• Thurston-Bennequin invariant
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• Rotation number
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• A knot is called **Legendrian simple** if its Legendrian isotopy classes are determined by Thurston-Bennequin and rotation numbers.
Mountain range

\[ r = \begin{array}{cccccc}
-2 & -1 & 0 & 1 & 2 \\
\end{array} \]

\[ tb = 1 \]

Right-handed trefoil
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\begin{align*}
0 \\
-1 \\
\end{align*}

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\[ m5_2 \]
Legendrian cables

• **Question**: If the classification of Legendrian knots of a given knot type is known, is it possible to classify Legendrian knots of its cables?
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• **(Tosun)** If $K$ is Legendrian simple, then $K_{p,q}$ is also Legendrian simple for $q/p > t_b(K) + 1$

• **(Chakraborty)** $\hat{\theta}(T^1) = \hat{\theta}(T^2)$ if and only if $\hat{\theta}(T^1_{p,q}) = \hat{\theta}(T^2_{p,q})$
Main result

Theorem (Chakraborty-Etnyre-M)
• For \( \frac{q}{p} > \overline{tb}(K) + 1 \), \( K_{p,q} \) is Legendrian simple if and only if \( K \) is Legendrian simple.
Main result

**Theorem (Chakraborty-Etnyre-M)**

- For $q/p > \overline{tb}(K) + 1$, $K_{p,q}$ is Legendrian simple if and only if $K$ is Legendrian simple.
- The mountain range of $K_{p,q}$ is a $(p, q)$-diamond of the mountain range of $K$. 
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\[ p \cdot r \]
\[ pq - |tp - q| \]
Examples

\[ r = \begin{array}{cccccc} & & & & & \\ & -2 & -1 & 0 & 1 & 2 \\ & & & & & \end{array} \]

\[ tb = 1 \]

\[ m_{5,2} \]
Examples

\[ r = \begin{array}{cccccc}
  \text{-2} & \text{-1} & 0 & 1 & 2 \\
 t_b = 1 & & & & &
\end{array} \]

\[ r = \begin{array}{cccccc}
  \text{-2} & \text{-1} & 0 & 1 & 2 \\
 t_b = 5 & 4 & 3 & & &
\end{array} \]

\[ m_{5_2} \]

\[ (2,3)\text{-cable of } m_{5_2} \]
Idea of Proof

• Put a cable $L_{p,q}$ on a standard neighborhood of $L$.
• Assume we have a common $L_{p,q}$ on neighborhoods of $L$ and $L'$.
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• There is a smooth isotopy from $N(L)$ to $N(L')$ fixing $L_{p,q}$
• Keep track of the contact structure on $N(L)$ during the isotopy
Thank you!