The contact mapping class group of lens spaces

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July 7, 2022

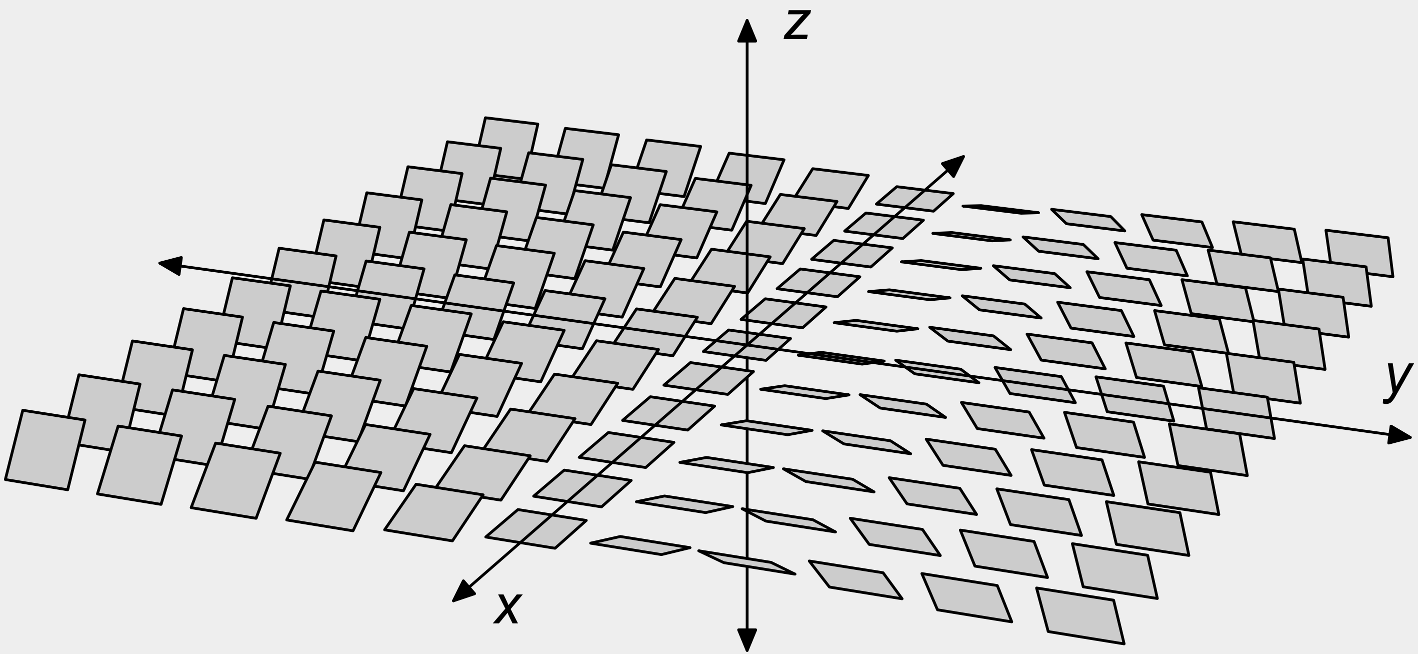
01. The contact mapping class group

Definitions

- A contact form of a 3-manifold M
- A contact structure of α

Examples

- The standard contact structure on $\mathbb{R}^3$



The standard contact structure $(\mathbb{R}^3, \xi_{std})$

Definitions

- A strict contactomorphism
- A coorientation preserving contactomorphism
- A coorientation reversing contactomorphism

Examples

- $\alpha_1 = dz - y\,dx, \;\; \alpha_2 = dz + x\,dy$
- coorientation preserving contactomorphism
- coorientation reserving contactomorphism

Today

- We only consider coorientation preserving self contactomorphisms
- We focus on contact structures, not contact forms
- Strict contactomorphisms depend on the choice of a contact form.
- Coorientation reversing contactomorphisms are confusing.

Definitions

- The group of contactomorphism : $\mathrm{Cont}(M, \xi)$

- The contact mapping class group: $\pi_0(\mathrm{Cont}(M, \xi))$

Exotic phenomena

- $i : \mathrm{Cont}(M, \xi) \to \mathrm{Diff}_+(M)$

- An exotic contactomorphism : $\ker i_* \neq 0$

Exotic phenomena

- $(S^1 \times S^2, \xi_{std})$
- f: a Dehn twist about $\{p\} \times S^2$
- (Gompf) $f^n \nsim f^m$ if $m \neq n$

The contact mapping class group

- Almost nothing is known

The contact mapping class group

The contact mapping class group

General strategy

- Fix a submanifold
- Determine the contact mapping class group of the complement

General strategy

- (S^3, ξ_{std})

02.

Lens spaces

Main theorem

$$\pi_0(\mathrm{Cont}(L(p,q),\xi_{std})) = \begin{cases} \mathbb{Z}_2 & p \neq 2 \text{ and } q \equiv -1 \pmod{p} \\ \mathbb{Z}_2 & q \not\equiv 1 \pmod{p} \text{ and } q^2 \equiv 1 \pmod{p} \\ 1 & \text{otherwise} \end{cases}$$

Strategy

- Classify Legendrian rational unknots in $L(p, q)$
  rational unknots: core of a Heegaard torus
- Perturb a contactomorphism to fix a neighborhood of a Legendrian rational unknot
- Determine the contact mapping class group of the complement.

Standard contact structures on $L(p,q)$

- $S^3 \subset \mathbb{C}^2$ $\quad \alpha_{std} = x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2$
- $L(p,q) = S^3/\mathbb{Z}_p \quad (z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi q i/p} z_2)$
- α_{std} is invariant under the $\mathbb{Z}_p$ action
  obtain an induced contact form α_{std} on $L(p,q)$

Standard contact structures on $L(p,q)$

- Also, obtain an induced contact form $-\alpha_{std}$ on $L(p,q)$
- (Giroux, Honda) On $L(p,q)$, there are
  2 universally tight contact structures if $q \not\equiv -1 \pmod{p}$
  1 universally tight contact structure if $q \equiv -1 \pmod{p}$
- $\xi_{std} \not\simeq -\xi_{std}$ if and only if $q \not\equiv -1 \pmod{p}$

Diffeomorphisms on $L(p, q)$

- $\sigma : L(p,q) \to L(p,q) \quad (z_1, z_2) \mapsto (z_2, z_1)$
  well defined if and only if $q^2 \equiv 1 \pmod{p}$

- $\tau : L(p,q) \to L(p,q) \quad (z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$

Contactomorphisms on $L(p, q)$

- $\sigma^*(\alpha_{std}) = \alpha_{std}$
  coorientation preserving

- $\tau^*(\alpha_{std}) = -\alpha_{std}$
  coorientation reversing

The mapping class group of $L(p,q)$

$$\pi_0(\mathrm{Diff}_+(L(p,q))) = \begin{cases} 1 & p = 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \langle \sigma, \tau \rangle & p \neq 2,\ q \not\equiv \pm 1 \ \text{and}\ q^2 \equiv 1 \pmod{p} \\ \mathbb{Z}_2 \cong \langle \tau \rangle & \text{otherwise} \end{cases}$$

Contact Morse theory

- Weinstein manifold

- Contact Morse function

Contact Morse theory

- It is useless

Contact Morse theory

- Characteristic foliation

- (Giroux) can perturb a surface to have a Morse+ characteristic foliation

Contact Morse theory

- (Giroux) We can perturb a contact structure on $\Sigma \times [0,1]$ so that
  - $(\Sigma \times \{t\})_\xi$ are Morse+ except for finite $t_1, \ldots, t_n$
  - $(\Sigma \times \{t_i\})_\xi$ are Morse
  - $\Sigma \times [t_i - \epsilon, t_i + \epsilon]$ is contactomorphic to a canceling pair of 1- and 2-handles.

Contact Morse theory

- (Colin) Giroux's theorem holds for a 1-parametric family of embedded surfaces in any contact 3-manifold.

- Still hard to use

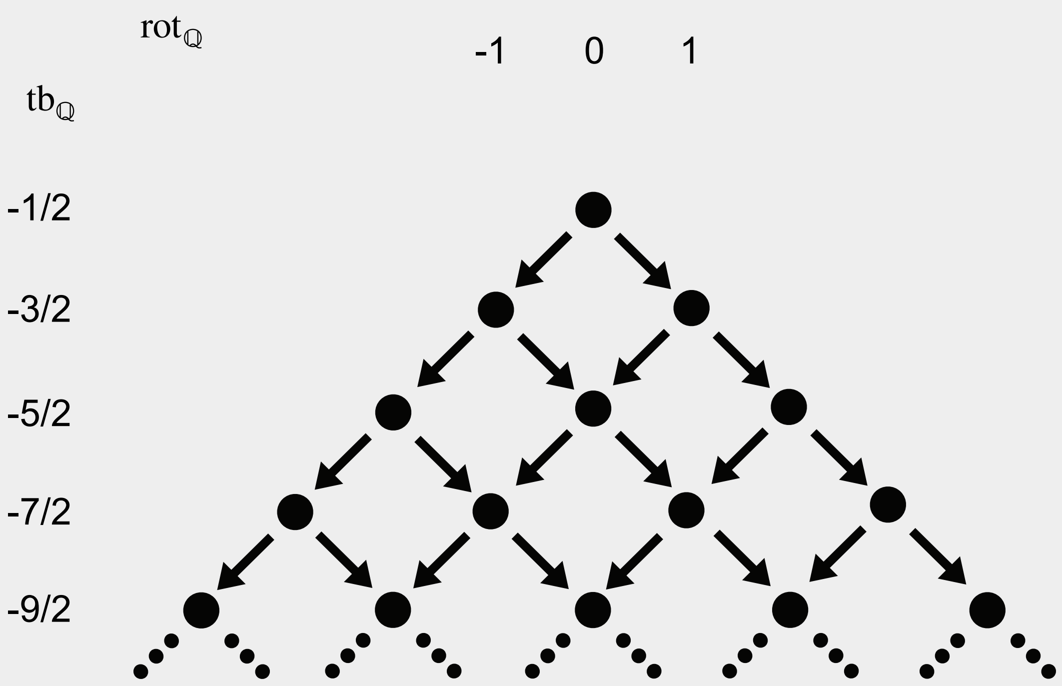
Contact mapping class group of $S^1 \times D^2$

- Universally tight + ϵ condition
- extremal relative Euler class on a meridian disk
- Any two meridian disks are contact isotopic

Contact mapping class group of $S^1 \times D^2$

- Fix a meridian disk

- Reduce the problem to a ball



Legendrian rational unknots in $(\mathbb{RP}^3, \xi_{std})$

Contact mapping class group of $\mathbb{RP}^3$

- (Bonahon) any contactomorphism f is smoothly isotopic to the identity
- L and $f(L)$ have the same $\mathrm{tb}_{\mathbb{Q}}$ $\Rightarrow$ L and $f(L)$ are Legendrian isotopic
- Contact isotopy extension theorem $\Rightarrow f$ fixes a neighborhood of L
- The complement of $N(L)$ is a universally tight solid torus

Thank you!