Non-loose torus knots

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01. Existence of non-loose knots
Question

• Which knot $K \subset M$ admits non-loose Legendrian representatives?
Theorem

• (Etnyre-M-Mukherjee) Let $M$ be an irreducible manifold. $K \subset M$ admits a non-loose Legendrian representative $\iff K$ is not contained in 3-ball or $M$ admits a tight contact structure.
Idea

• (Colin) Let \((M, \xi)\) is a contact 3-manifold. If \(M = M_1 \cup_{S^2} M_2\) and \(\xi|_{M_1}, \xi|_{M_2}\) are tight, then \(\xi\) is tight.

• If \(M\) does not admit a tight contact structure, then \(M \setminus B^3\) does not admit tight contact structure.

• \(K \subset B^3 \subset M\) is always loose.
Corollary

• Any knot $K \subset S^3$ admits a non-loose Legendrian representative.

• Any knot $K \subset \Sigma(2,3,-5)$ contained in a ball does not admit a non-loose Legendrian representative.
02. Non-loose torus knots
Classification of Legendrian knots

- Goal: classify non-loose Legendrian knots up to contactomorphism.

- \( \pi_0(\text{Cont}(S^3, \xi_i)) \neq 0 \)
Invariants of non-loose knots

- (extended) classical invariants: \((tb, rot, torsion, \xi_i)\)
- \(tb\): Thurston-Bennequin invariant
- \(rot\): rotation number
- \(torsion\): # of boundary-parallel Giroux torsions in \(S^3 \setminus N(K)\)
- \(\xi_i\): the ambient contact structure
Giroux torsions

• Giroux torsion $i$
  \[(T^2 \times I, \cos(2i\pi t)dx - \sin(2i\pi t) dy)\]

• Giroux torsion $i + \frac{1}{2}$
  \[(T^2 \times I, \cos((2i + 1)\pi t)dx - \sin((2i + 1)\pi t) dy)\]
Mountain range

- Fix torsion and $\xi_i$
- $x$-coordinate: $\text{rot}$
- $y$-coordinate: $tb$
- Each dot represents a non-loose Legendrian representative.
- Arrows indicate stabilizations
Right-handed trefoil in \((S^3, \xi_{\text{std}})\)
Classification of Legendrian knots

- Classification = completing the mountain range.
Previous results

**Unknot (Eliashberg-Fraser)**

- Every non-loose unknot is contained in $(S^3, \xi_1)$ and has 0 torsion.

- For $i \geq 1$, there exists a family of non-loose Legendrian unknot $\{L_i^\pm\}$ with

\[
(tb, rot, torsion) = (i, \pm (i - 1), 0)
\]
Non-loose unknots in $(S^3, \xi_1)$
Previous results

Left-handed trefoil (Geiges-Onaran)

- classified non-loose left-handed trefoils with torsion 0 and $tb < -6$ and $tb = -5$
- For $n < -6$, there exist two non-loose Legendrian representatives with $tb = n$, $rot = \pm(n - 1)$
Non-loose LHT in $(S^3, \xi_2)$ with torsion 0
Previous results

Right-handed trefoil (Geiges-Onaran)

- Classified non-loose right-handed trefoils with 0 torsion and $tb = 7$.
- There exist four non-loose Legendrian representatives with $rot = \pm 4, \pm 8$. 
Non-loose RHT with $tb = 7$ and torsion 0
Previous results

**Negative torus knots (Matkovic)**

- # of non-loose $(p,q)$-torus knot with $tb < pq$ and torsion 0 $(q < 0)$

  $$= \text{# of tight contact structures on } S^3_{tb-\frac{1}{2}}(T_p,-q)$$
Theorem

• (Etnyre-M-Mukherjee) classified non-loose torus knots
Non-loose LHT in $(S^3, \xi_2)$ with torsion $= 0$
Non-loose LHT in $(S^3, \xi_2)$ with torsion $i > 0$
Non-loose LHT in \((S^3, \xi_1)\) with torsion \(i+1/2\)
Non-loose RHT in \((S^3, \xi_0)\)
Non-loose RHT in $\left( S^3, \xi_{-1} \right)$
Non-loose RHT in $\left( S^3, \xi_1 \right)$ with torsion 0
Theorem (cont.)

• There exist $m(p, q)$ non-loose torus knots with $tb = pq$ with torsion 0

• $m(p, q) = \#\text{Tight} \left( S^1 \times D^2, \frac{q}{p} \right) \cdot \#\text{Tight} \left( S^1 \times D^2, \frac{p}{q} \right)$
A decomposition of $S^3$
Surgery diagram for \( tb=pq \)
Two types of non-loose Legendrian

• $d(L) = \text{minimum intersection number of } L \text{ and an overtwisted disk}$

• inconsistent Legendrian: $d(L) = 1$

• consistent Legendrian: $d(L) > 1$
Theorem

Let $L$ be a non-loose negative torus knot. If $L$ is

1. inconsistent, then
   • it can have any contact framing ($tb$)
   • adding a torsion, it is still non-loose.

2. consistent, then
   • $\bar{tb} = pq$,
   • adding a torsion, it becomes loose.
Non-loose $T(5,-8)$ in $\xi_2$ with torsion 0
Theorem (cont.)

Let $L$ be a non-loose positive torus knot. $L$ is either
1. inconsistent, or
2. consistent, or
3. cables of non-loose unknot ($\xi_1$).
RHT as (2,3) cable of the unknot
Applications

• (Etnyre-M-Tosun, work in progress)
  Tight contact structures on $S_r^3(T_{p,q})$

• $Tight \left( S_r^3(T_{p,q}) \right) = \text{surgery from } T_{p,q} \subset (S^3, \xi_{std})$
  + surgery from non-loose $T_{p,q}$
Thank you!

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