Non-loose torus knots

Hyunki Min
MIT

G & T Workshop Turkey
December 8, 2021

01.

Existence of non-loose knots

Question

- Which knot $K \subset M$ admits non-loose Legendrian representatives?

Theorem

- (Etnyre-M-Mukherjee) Let M be an irreducible manifold.

$K \subset M$ admits a non-loose Legendrian representative
$$\Leftrightarrow$$
K is not contained in 3-ball or M admits a tight contact structure.

Idea

- (Colin) Let (M, ξ) is a contact 3-manifold.
  If $M = M_1 \cup_{S^2} M_2$ and $\xi|_{M_1}, \xi|_{M_2}$ are tight, then ξ is tight.

- If M does not admit a tight contact structure, then
  $M \setminus B^3$ does not admit tight contact structure.

- $K \subset B^3 \subset M$ is always loose.

Corollary

- Any knot $K \subset S^3$ admits a non-loose Legendrian representative.
- Any knot $K \subset \Sigma(2,3,-5)$ contained in a ball does not admit a non-loose Legendrian representative.

02.

Non-loose torus knots

Classification of Legendrian knots

- Goal: classify non-loose Legendrian knots up to contactomorphism.
- $\pi_0\big(Cont(S^3, \xi_i)\big) \neq 0$

Invariants of non-loose knots

- (extended) classical invariants: $(tb, rot, torsion, \xi_i)$

- tb: Thurston-Bennequin invariant
- rot: rotation number
- $torsion$: # of boundary-parallel Giroux torsions in $S^3 \setminus N(K)$
- ξ_i: the ambient contact structure

Giroux torsions

- Giroux torsion i

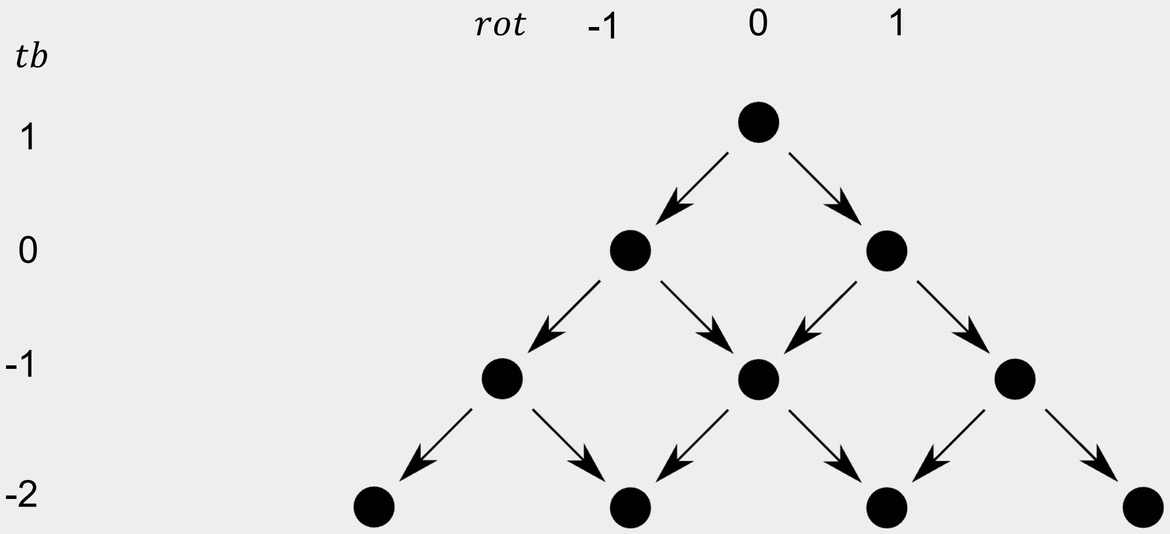
$$(T^2 \times I, \cos(2i\pi t)dx - \sin(2i\pi t)\,dy)$$

- Giroux torsion $i + \frac{1}{2}$

$$\left(T^2 \times I, \cos\big((2i+1)\pi t\big)dx - \sin\big((2i+1)\pi t\big)\,dy\right)$$

Mountain range

- Fix $torsion$ and ξ_i
- x-coordinate: rot
- y-coordinate: tb
- Each dot represents a non-loose Legendrian representative.
- Arrows indicate stabilizations



Right-handed trefoil in (S^3, ξ_{std})

Classification of Legendrian knots

- Classification = completing the mountain range.

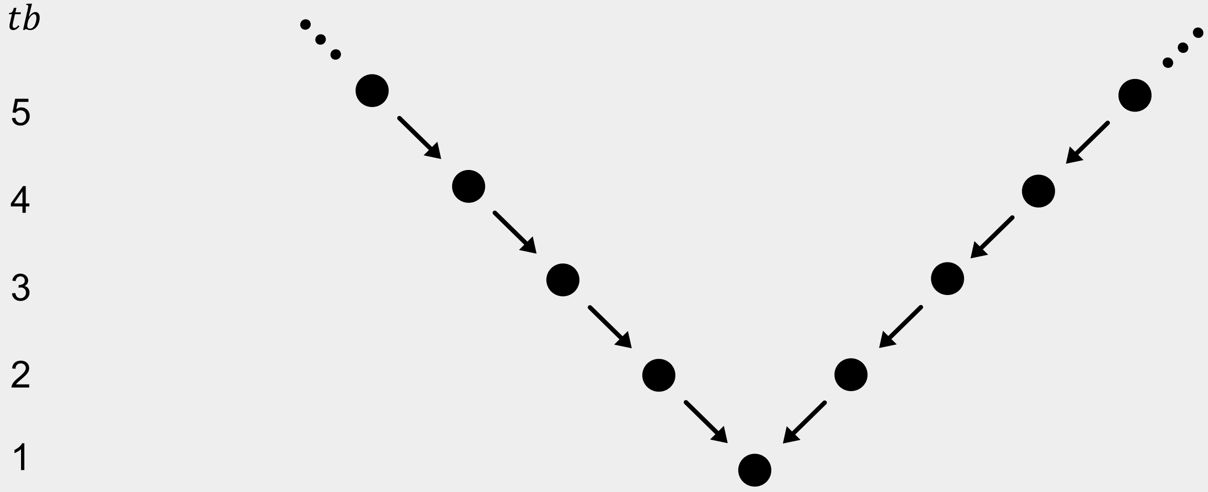
Previous results

Unknot (Eliashberg-Fraser)

- Every non-loose unknot is contained in (S^3, ξ_1) and has 0 torsion.

- For $i \geq 1$, there exists a family of non-loose Legendrian unknot $\{L_i^{\pm}\}$ with

$$(tb, rot, torsion) = (i, \pm(i-1), 0)$$

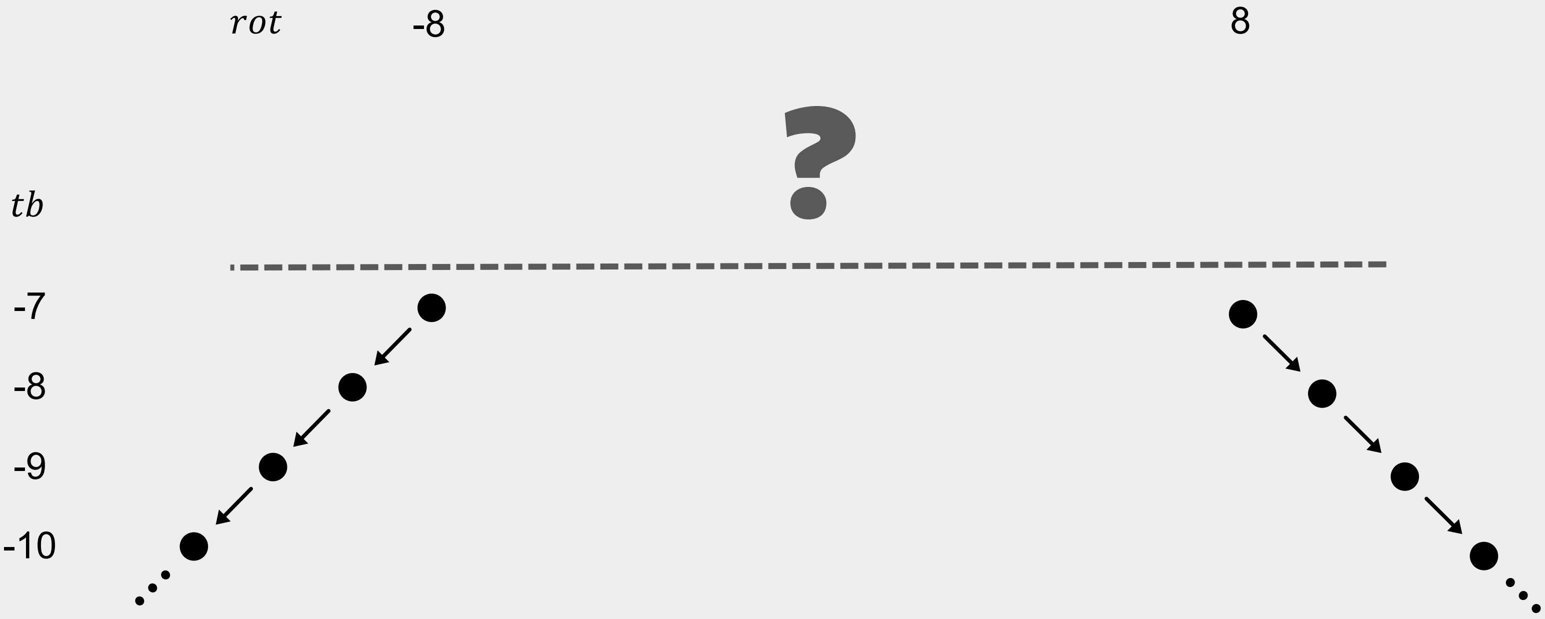


Non-loose unknots in $(S3, \xi_1)$

Previous results

Left-handed trefoil (Geiges-Onaran)

- classified non-loose left-handed trefoils with torsion 0 and tb < -6 and $tb = -5$
- For $n < -6$, there exist two non-loose Legendrian representatives with $tb = n, rot = \pm(n-1)$

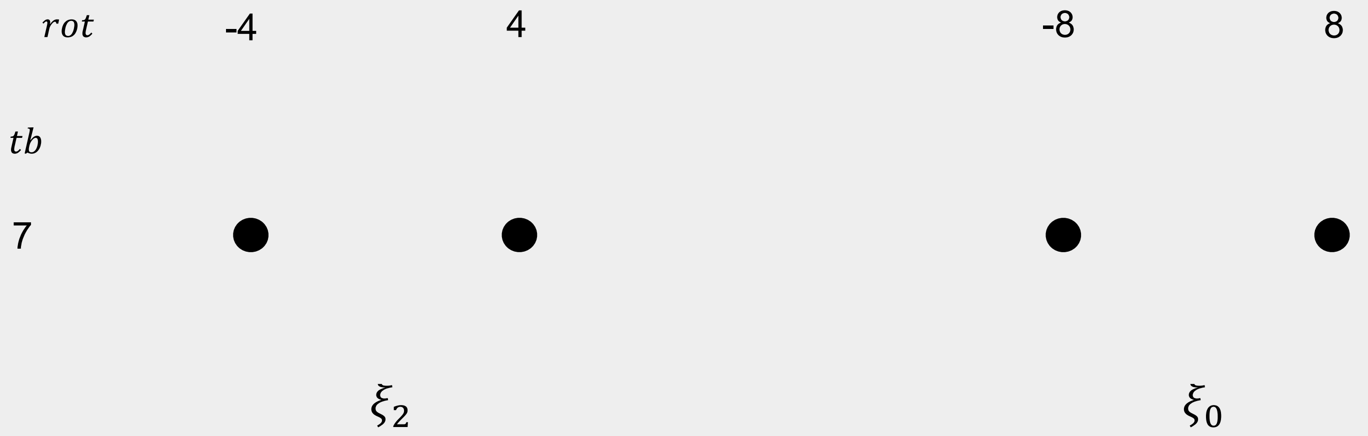


Non-loose LHT in (S^3, ξ_2) with torsion 0

Previous results

Right-handed trefoil (Geiges-Onaran)

- Classified non-loose right-handed trefoils with 0 torsion and $tb = 7$.
- There exist four non-loose Legendrian representatives with $rot = \pm 4, \pm 8$.



Non-loose RHT with $tb = 7$ and torsion 0

Previous results

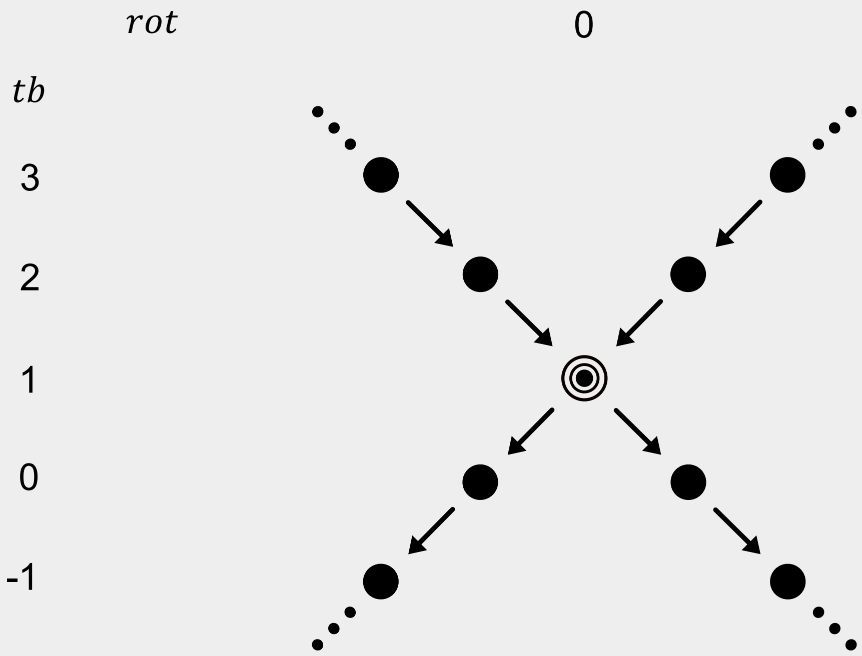
Negative torus knots (Matkovic)

- # of non-loose (p,q)-torus knot with $tb < pq$ and torsion 0 $(q < 0)$

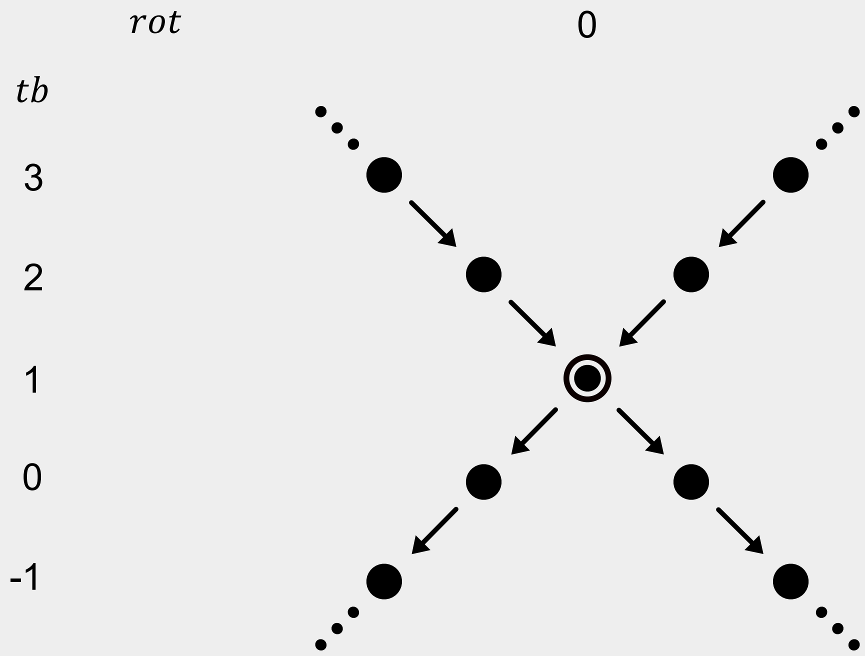
  = # of tight contact structures on $S^3_{tb-\frac{1}{2}}(\mathrm{T}_{\mathrm{p},-\mathrm{q}})$

Theorem

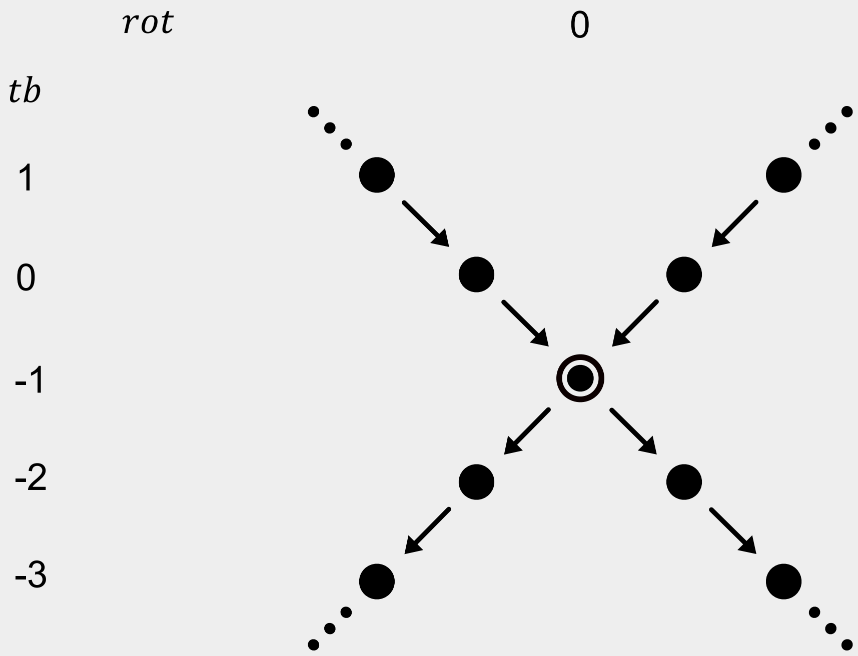
- (Etnyre-M-Mukherjee)
  classified non-loose torus knots



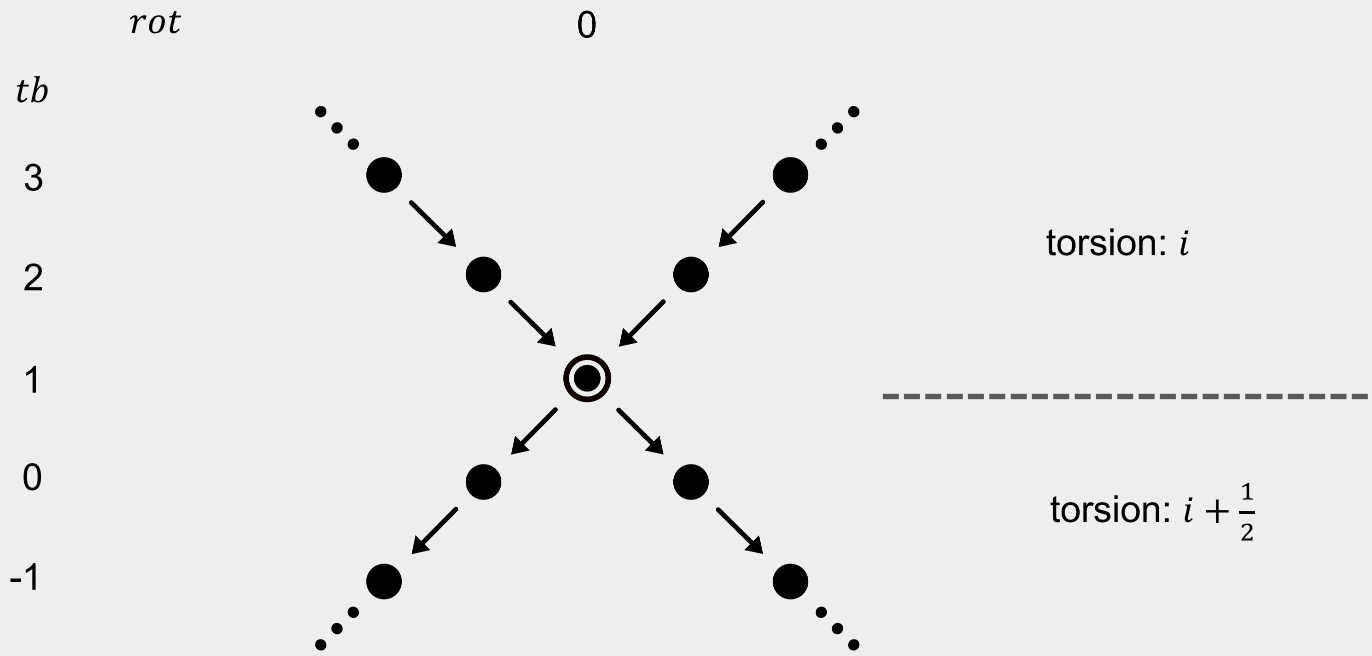
Non-loose LHT in (S^3, ξ_2) with torsion = 0



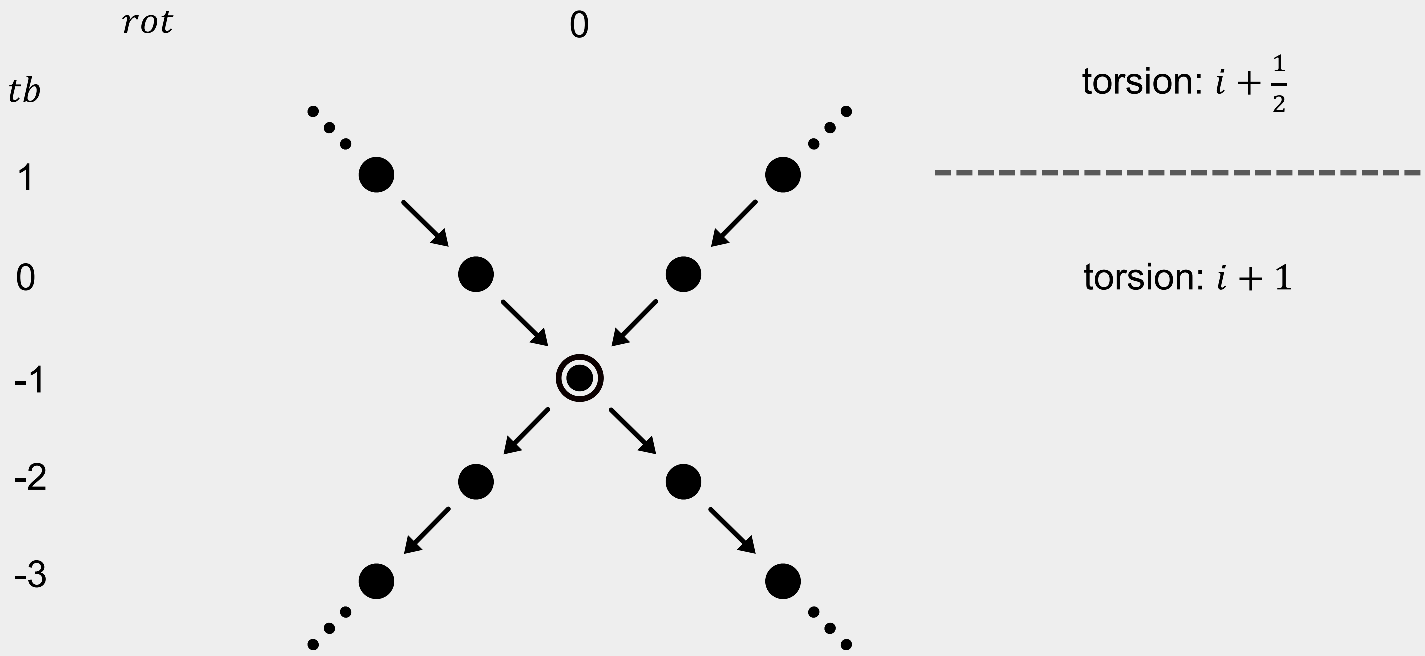
Non-loose LHT in (S^3, ξ_2) with torsion i > 0



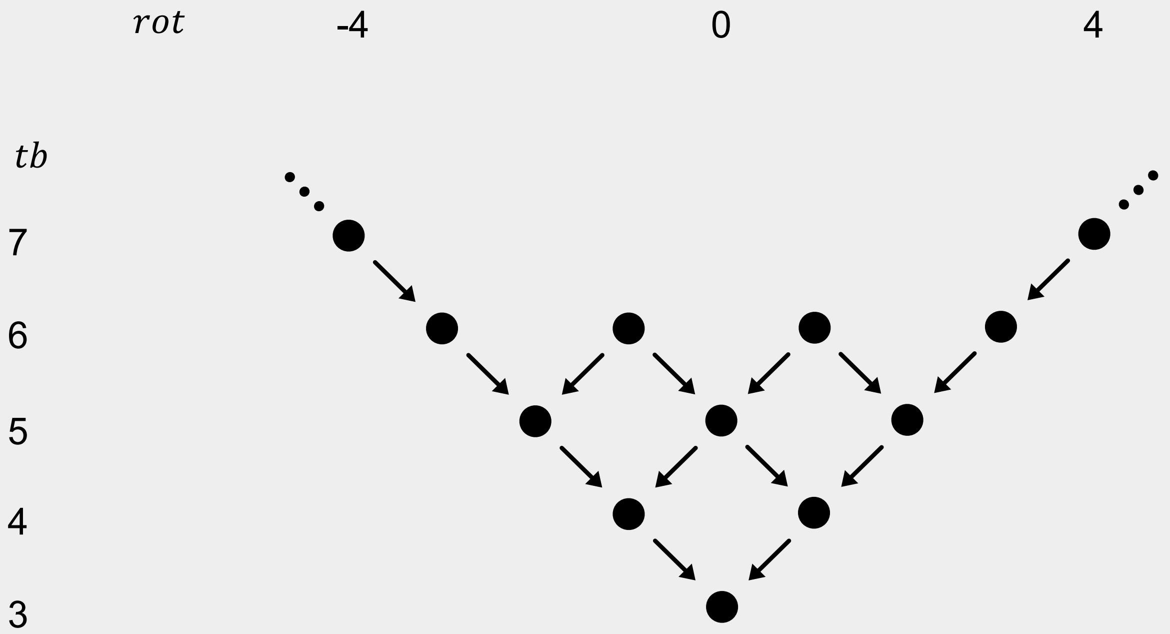
Non-loose LHT in (S^3, ξ_1) with torsion i+1/2



Non-loose RHT in (S^3, ξ_0)



Non-loose RHT in (S^3, ξ_{-1})

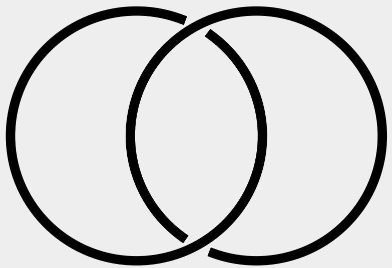


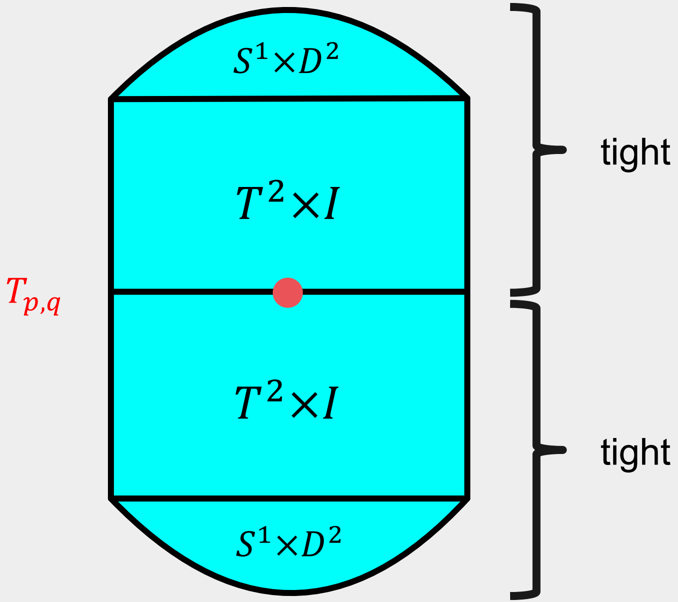
Non-loose RHT in (S^3, ξ_1) with torsion 0

Theorem (cont.)

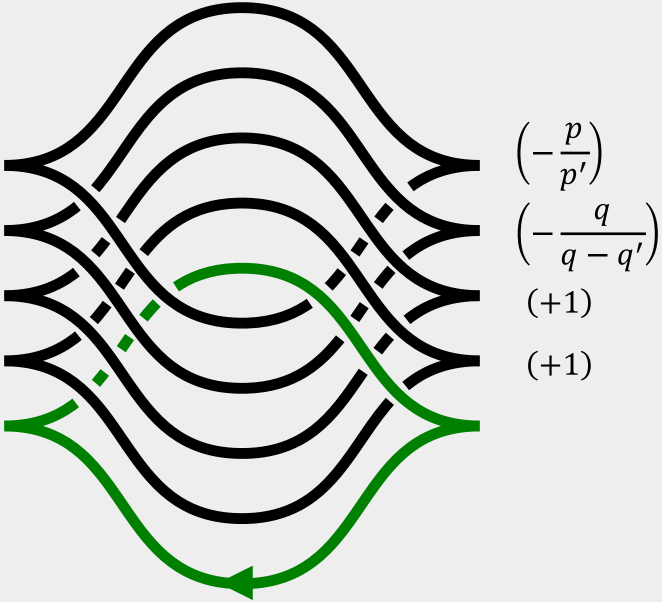
- There exist $m(p,q)$ non-loose torus knots with $tb = pq$ with torsion 0

- $m(p,q) = \#Tight\left(S^1 \times D^2, \frac{q}{p}\right) \cdot \#Tight\left(S^1 \times D^2, \frac{p}{q}\right)$





A decomposition of S^3



Surgery diagram for tb=pq

Two types of non-loose Legendrian

- $d(L)$ = minimum intersection number of L and an overtwisted disk

- inconsistent Legendrian: $d(L) = 1$

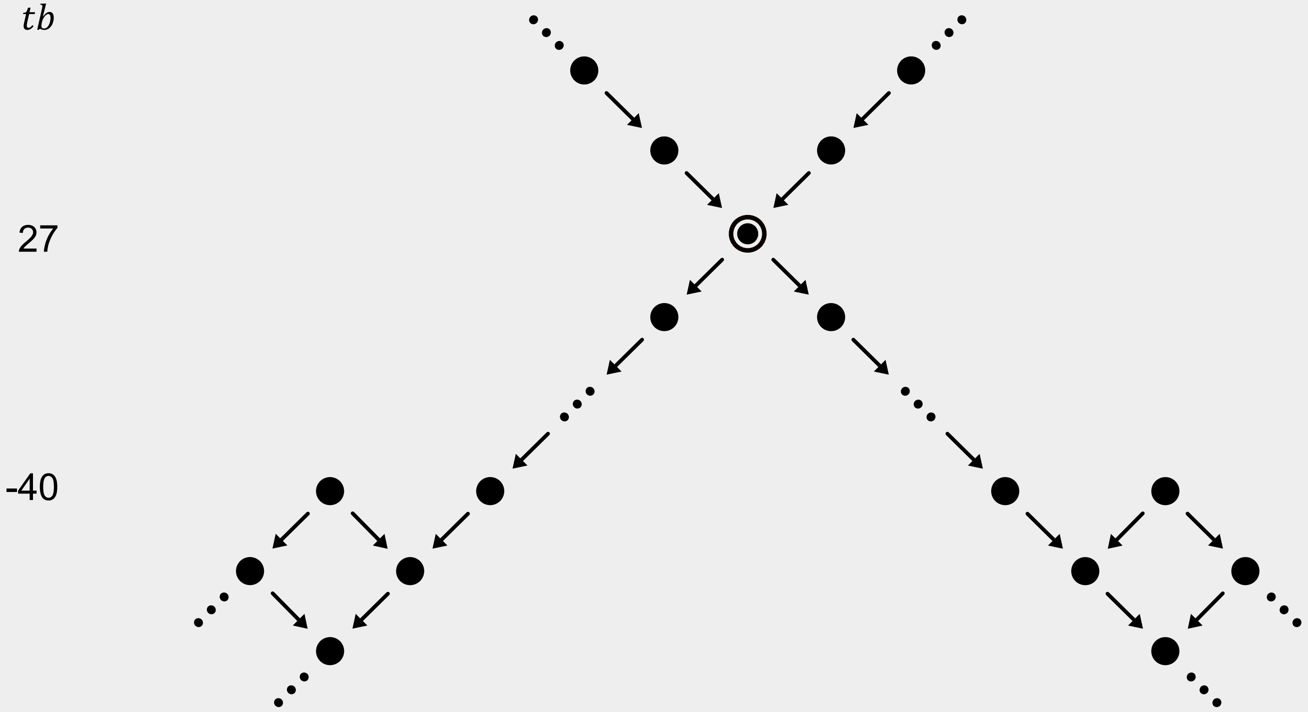
- consistent Legendrian: $d(L) > 1$

Theorem

Let L be a non-loose negative torus knot. If L is

1. inconsistent, then
   - it can have any contact framing (tb)
   - adding a torsion, it is still non-loose.

2. consistent, then
   - $\overline{tb} = pq$,
   - adding a torsion, it becomes loose.



Non-loose T(5,-8) in ξ_2 with torsion 0

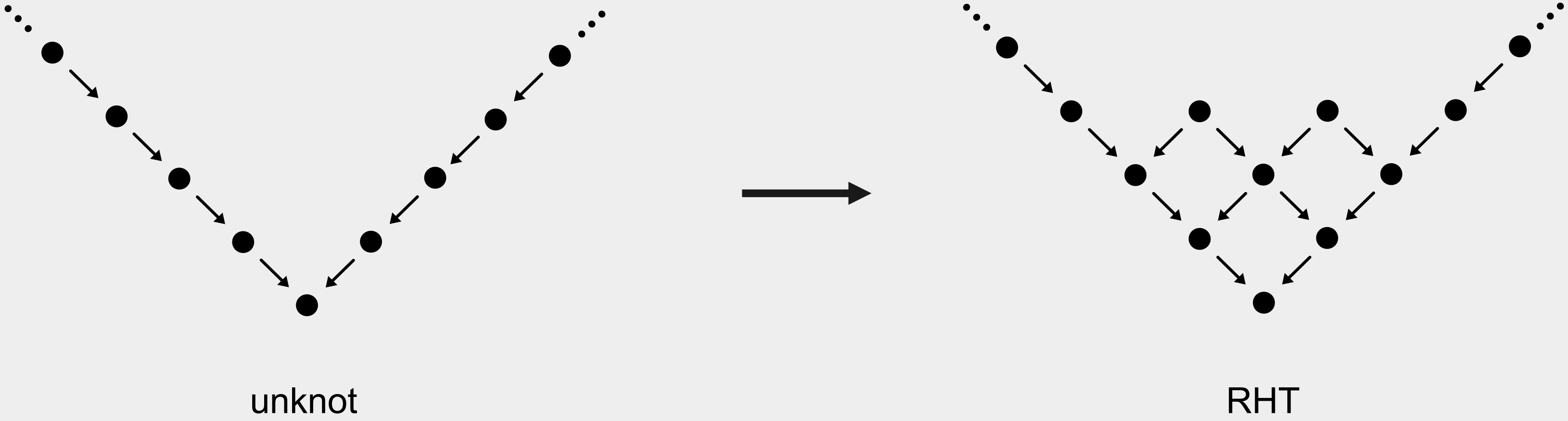
Theorem (cont.)

Let L be a non-loose positive torus knot. L is either

1. inconsistent, or

2. consistent, or

3. cables of non-loose unknot (ξ_1).



RHT as (2,3) cable of the unkot

Applications

- (Etnyre-M-Tosun, work in progress)
  Tight contact structures on $S^3_r(T_{p,q})$

- $Tight\left(S^3_r(T_{p,q})\right)$ = surgery from $T_{p,q} \subset (S^3, \xi_{std})$
  + surgery from non-loose $T_{p,q}$

Thank you!