Spinal open books and symplectic fillings

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Open book and Symplectic fillings

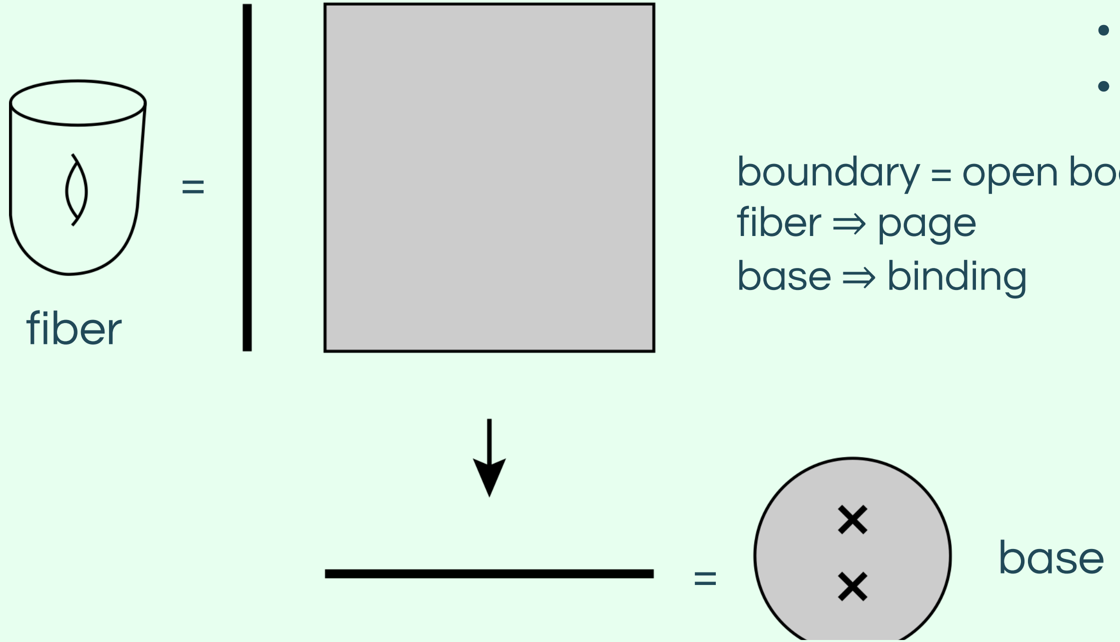
Theorem (Wendl)

A symplectic filling of a planar open book is supported by a Lefschetz fibration

Theorem (Lisi-VHM-Wendl) (Min-Roy-Wang)

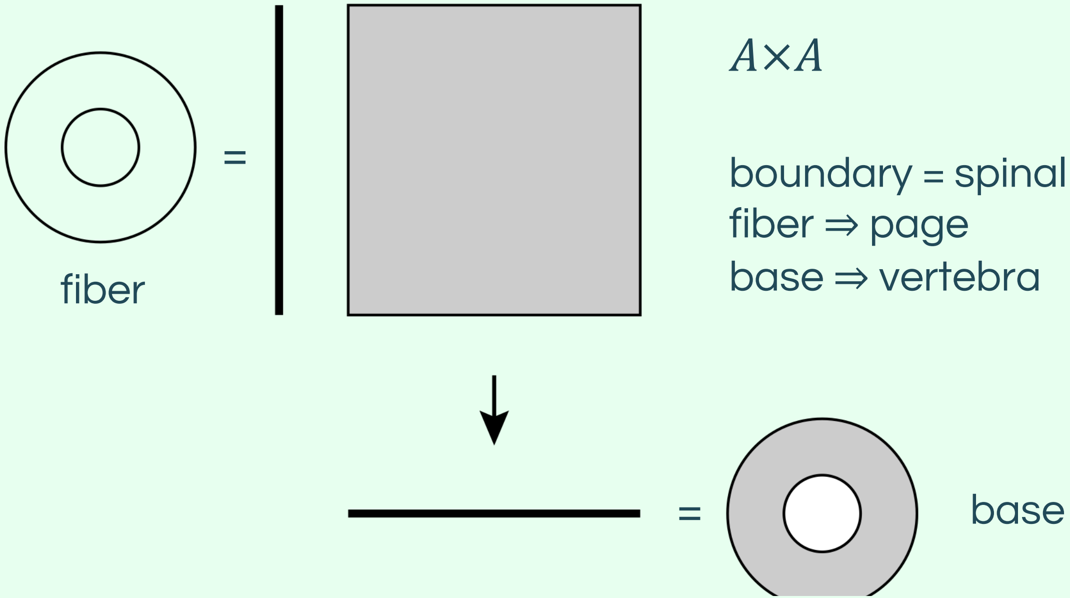
A symplectic filling of a planar uniform spinal open book is supported by a nearly Lefschetz fibration

Lefschetz fibrations over a disk



boundary = open book
fiber ⇒ page
base ⇒ binding

Lefschetz fibrations over a surface



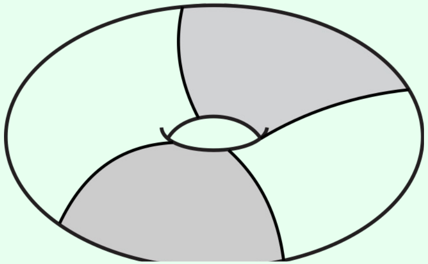
$A \times A$

boundary = spinal open book
fiber ⇒ page
base ⇒ vertebra

Spinal open book decompositions

$$A \times A = (S^1 \times I) \times (S^1 \times I)$$
$$= (S^1 \times S^1) \times (I \times I)$$
$$= T^2 \times D^2$$

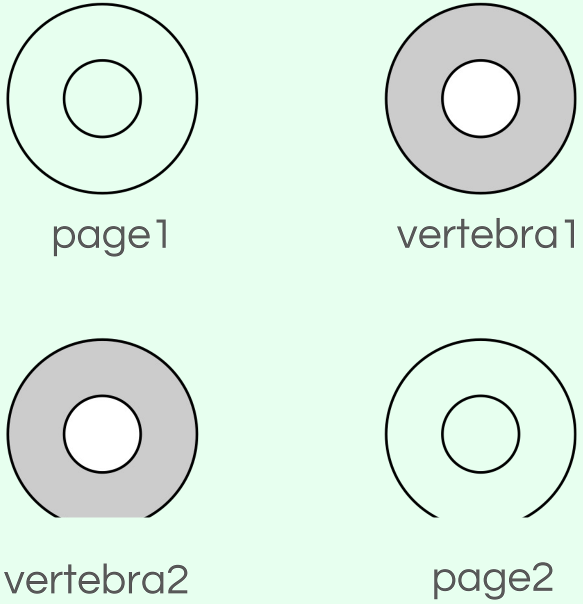
$$\Rightarrow \partial(A \times A) = T^2 \times S^1 = T^3$$



$\times S^1$

Spinal open book decompositions

A (Lefschetz amenable) spinal open book for T^3



Spinal open book decompositions

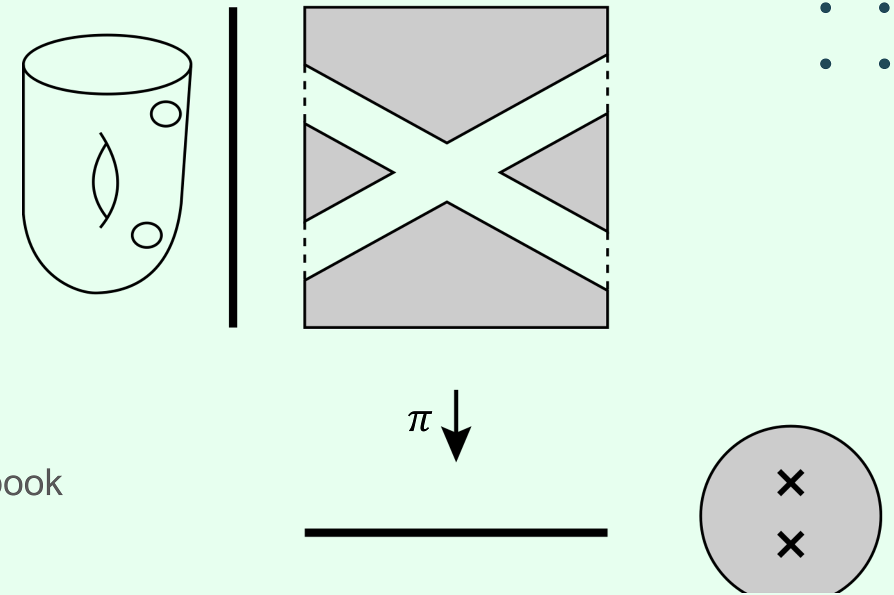
- $M = M_P \cup M_\Sigma$; M_P: paper, M_Σ: spine, P: page, Σ: vertebra
- $\pi = (\pi_\Sigma, \pi_P)$
- $\pi_\Sigma: M_\Sigma \to \Sigma$ is a trivial S^1 bundle over Σ
- $\pi_P: M_P \to S^1$ is a P-bundle over S^1
- A fiber of M_Σ is identified with ∂P

Nearly Lefschetz fibrations

complement of a **multisection** of the base

multisection: branched cover of the base

Boundary: a uniform spinal open book



Question: How to construct a spinal open book?

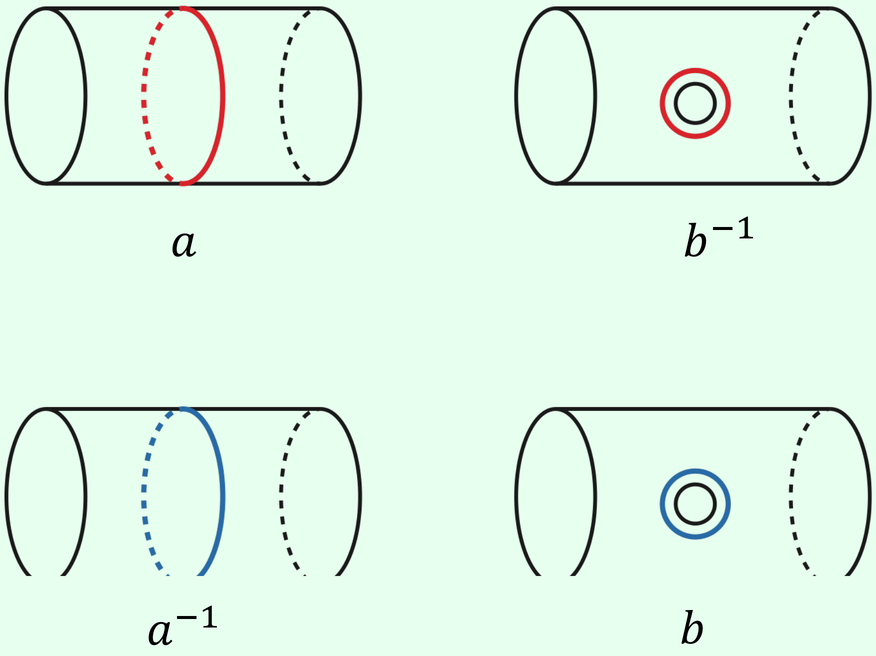
Method 1 Use relative open books

Method 2 Quasipositive braids

Relative open books

- Parts of an honest open book.
- Building blocks for an honest open book

Relative open books for $T^2 \times I$



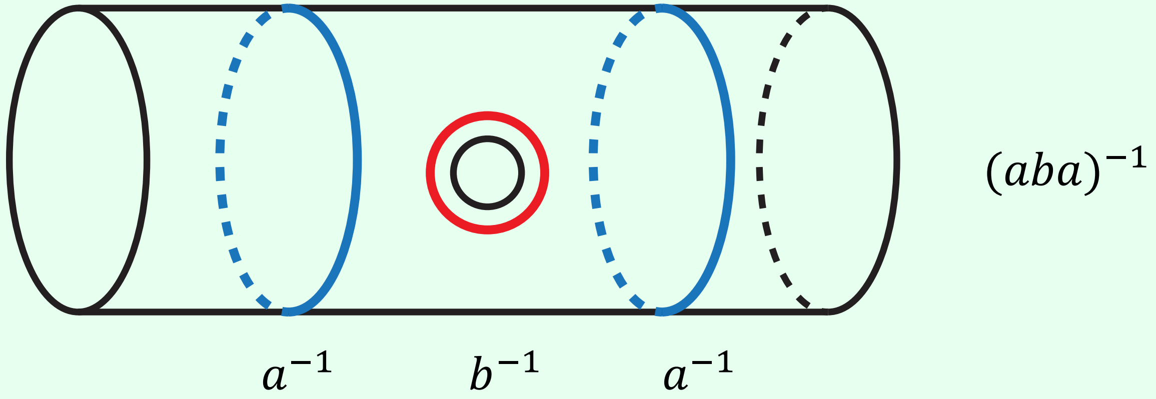
Torus bundles

- A: 2x2 matrix
- $T_A = T \times [0,1]/\sim \quad (x,1) \sim (Ax,0)$
- Example: $T^3 = T_I$ parabolic: $A = \pm \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

Torus bundles

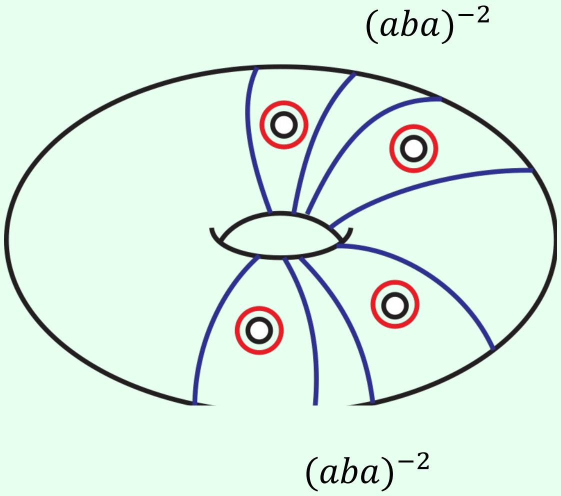
- $a \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ b $\sim \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
- $(aba)^{-2} \sim -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $(aba)^{-4} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Relative open books



Relative open books

- $(aba)^{-2} \cong -\mathrm{I}$
- $(aba)^{-4} \cong \mathrm{I}$



An open book for T^3

Relative open books

Theorem (VHM)

The relative open book with the word $(aba)^{-2}$ is contactomorphic to a π-twisting $T^2 \times I$

Theorem (Min-Roy-Wang)

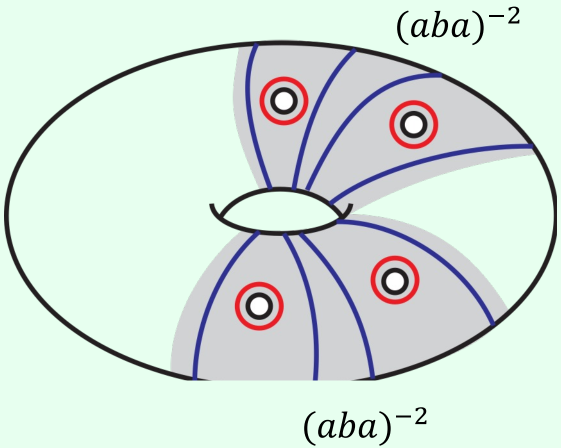
The spine from an annular vertebra is contactomorphic to a π-twisting $T^2 \times I$

From an open book to a spinal open book

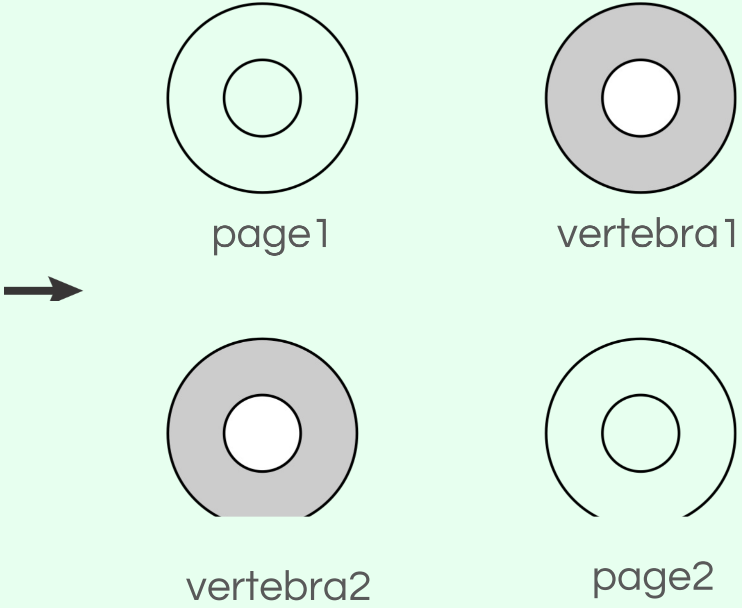
Find a relative open book with the word $(aba)^{-2}$

Remove the relative open book and replace it with an annulus vertebra

Exmaple: T^3

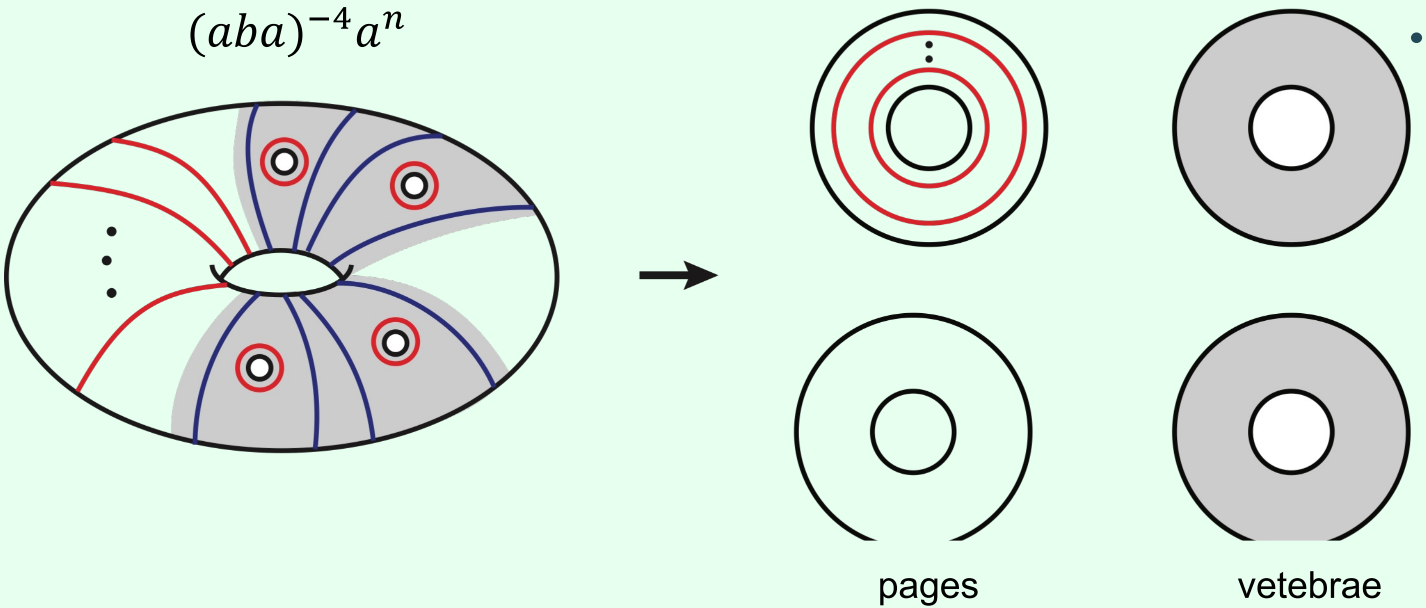


An open book for T^3



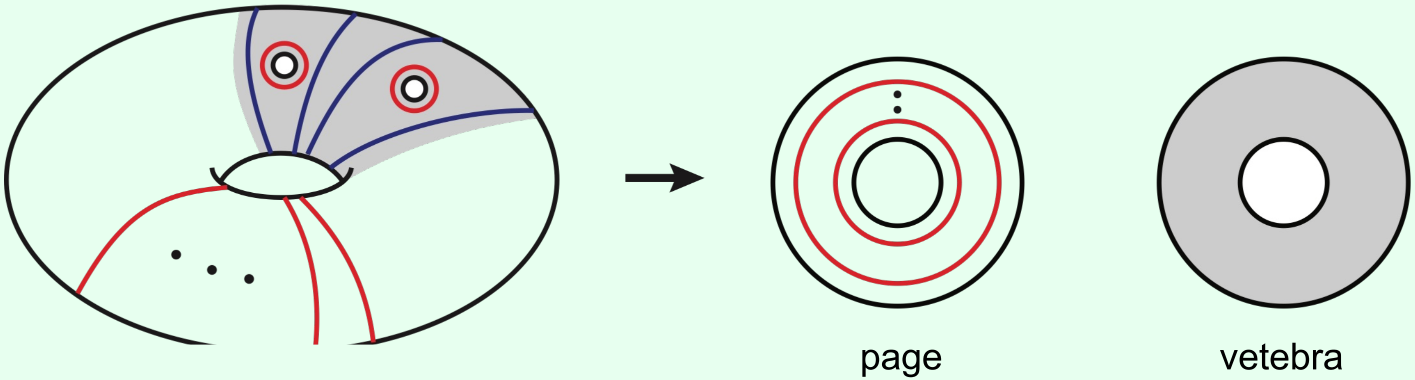
A spinal open book for T^3

Positive parabolic torus bundles



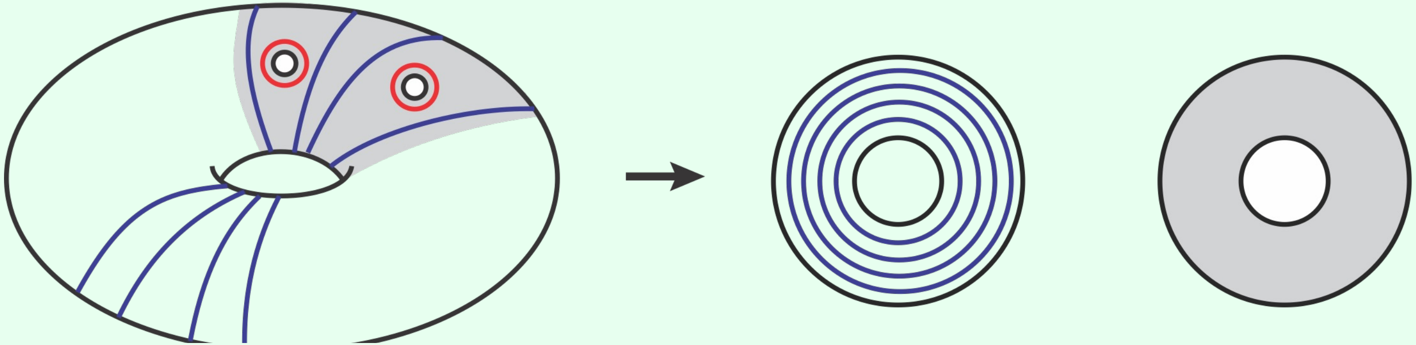
Negative parabolic torus bundles

$(aba)^{-2}a^n$



Negative parabolic torus bundles

$(aba)^{-2}a^{-4}$



Negative parabolic torus bundles

Theorem
(VHM)
(Golla-Lisca)
(Ding-Li)

$(aba)^{-2}a^n$ is symplectically fillable if and only if $n \geq -4$

Theorem
(Min-Roy-Wang)

$(aba)^{-2}a^n$ admits a unique Stein filling if and only if $n \geq -4$

Negative parabolic torus bundles

Theorem

For $n \geq 6$, $S^3_n(T_{2,3})$ admits two tight contact structures

Theorem (Min-Roy-Wang)

For $n \geq 6$, each tight contact structure on $S^3_n(T_{2,3})$ admits a unique symplectic filling

Thank you!